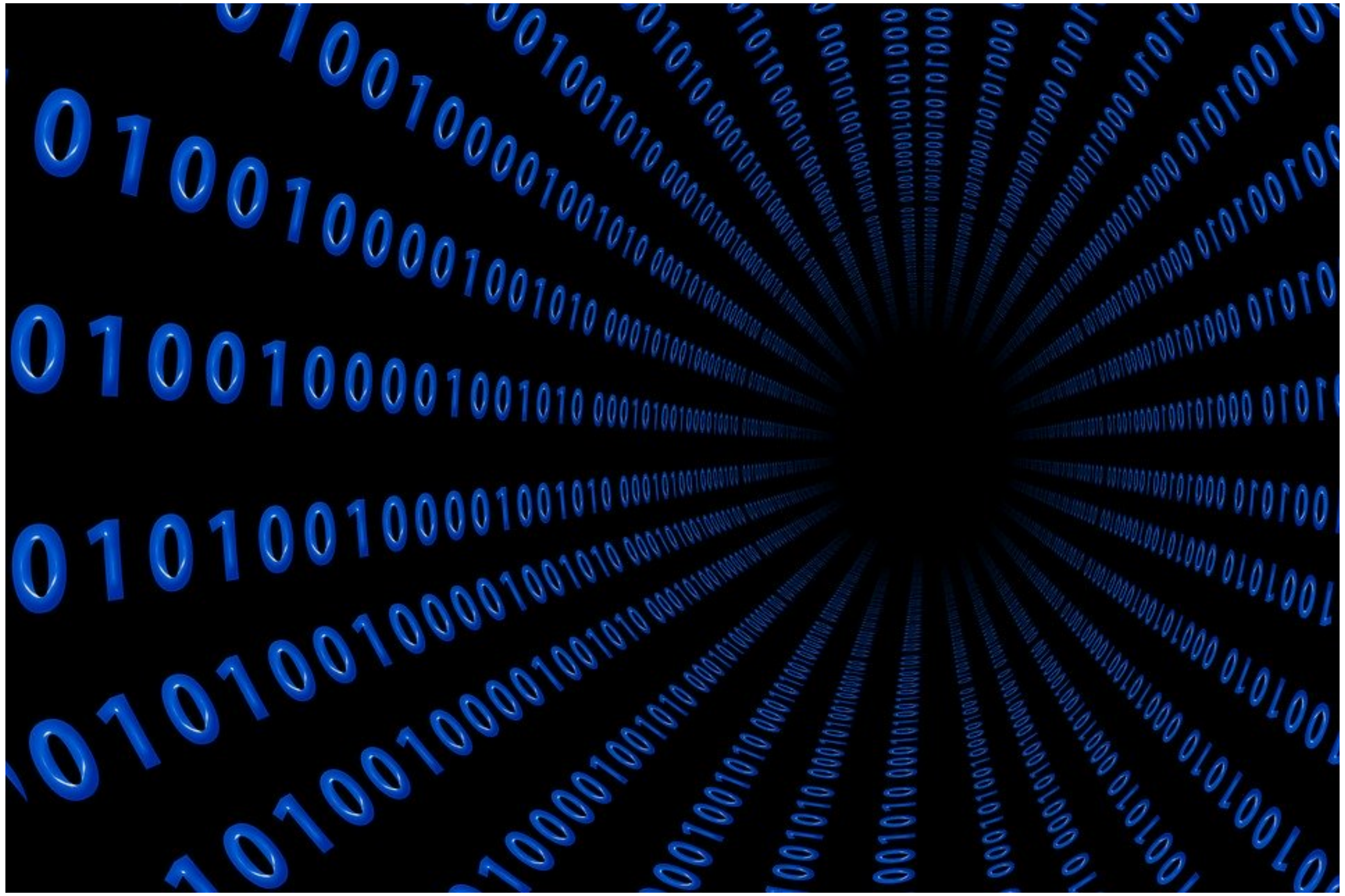
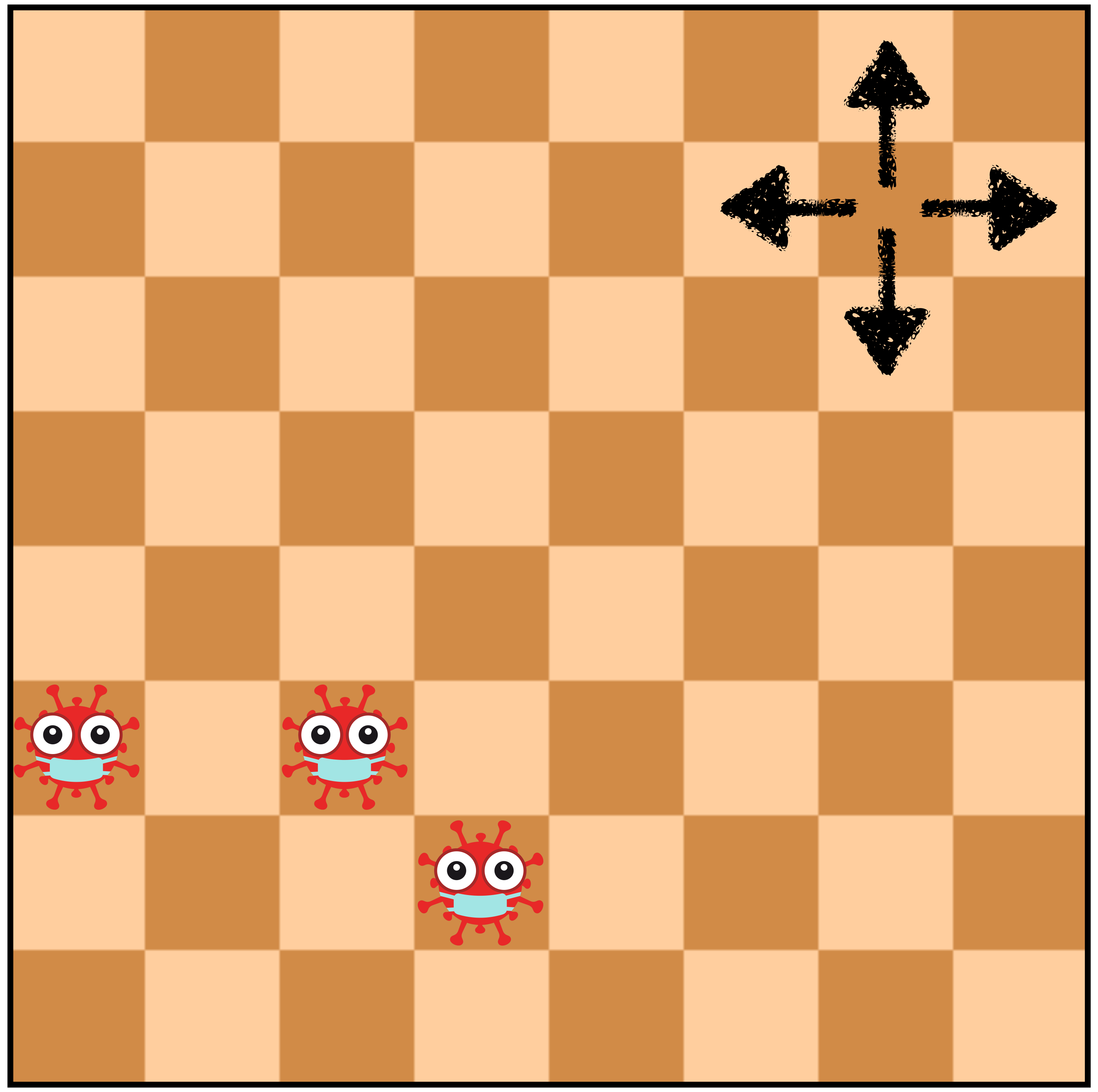
CS251

Great Ideas in *Theoretical* Computer Science



Strings, Encodings, Problems

Covid Puzzle



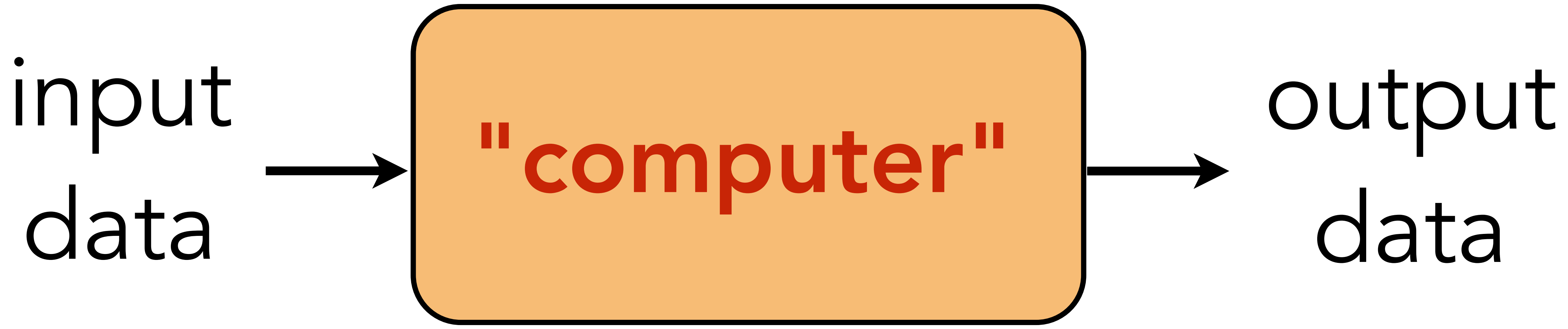
Neighbors in directions **N**, **S**, **W**, **E.**

Initially, some of the squares are **infected**.

If a square has 2 or more **infected** neighbors, it becomes **infected**.

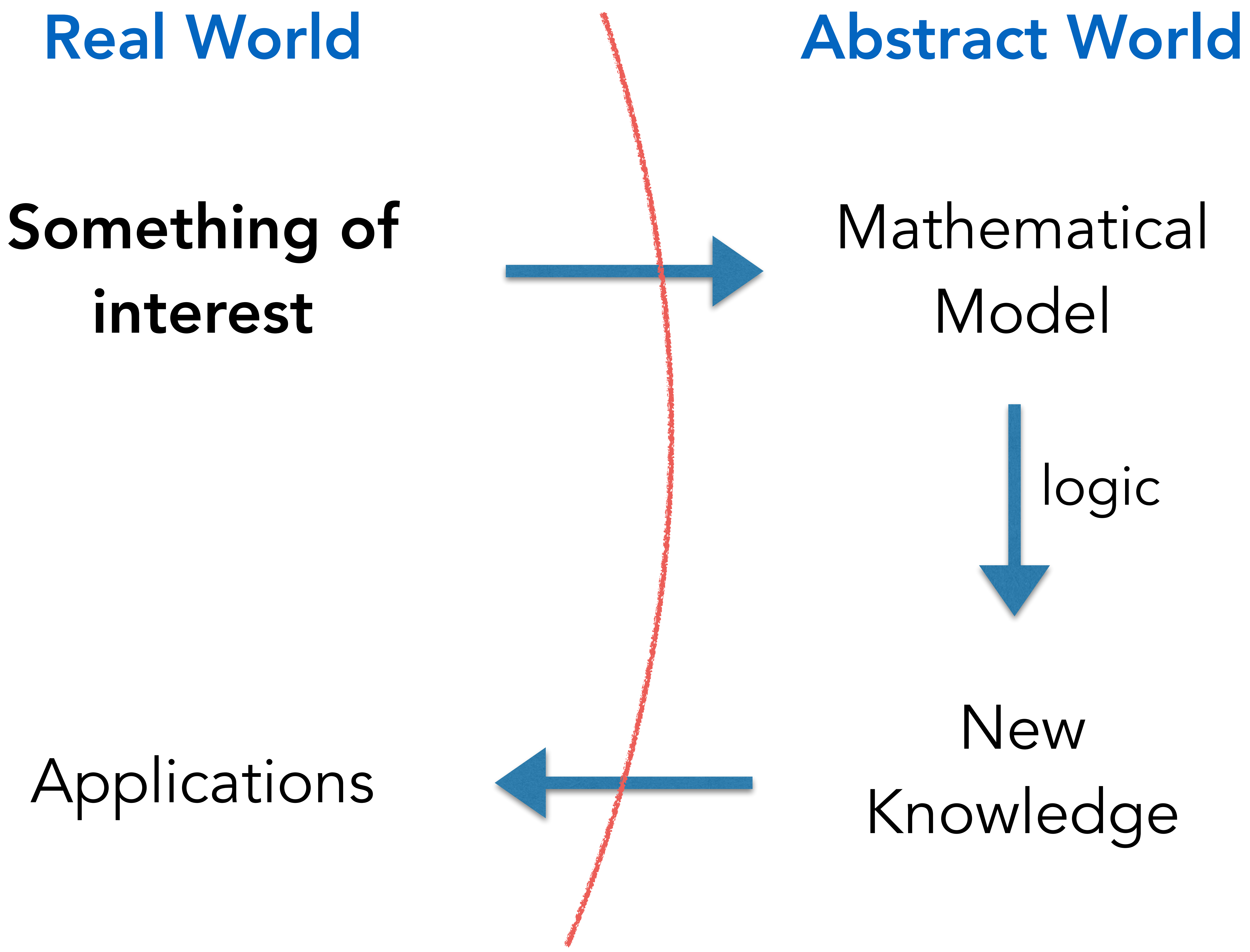
What is the min number of **infected** squares needed initially to infect the whole board?

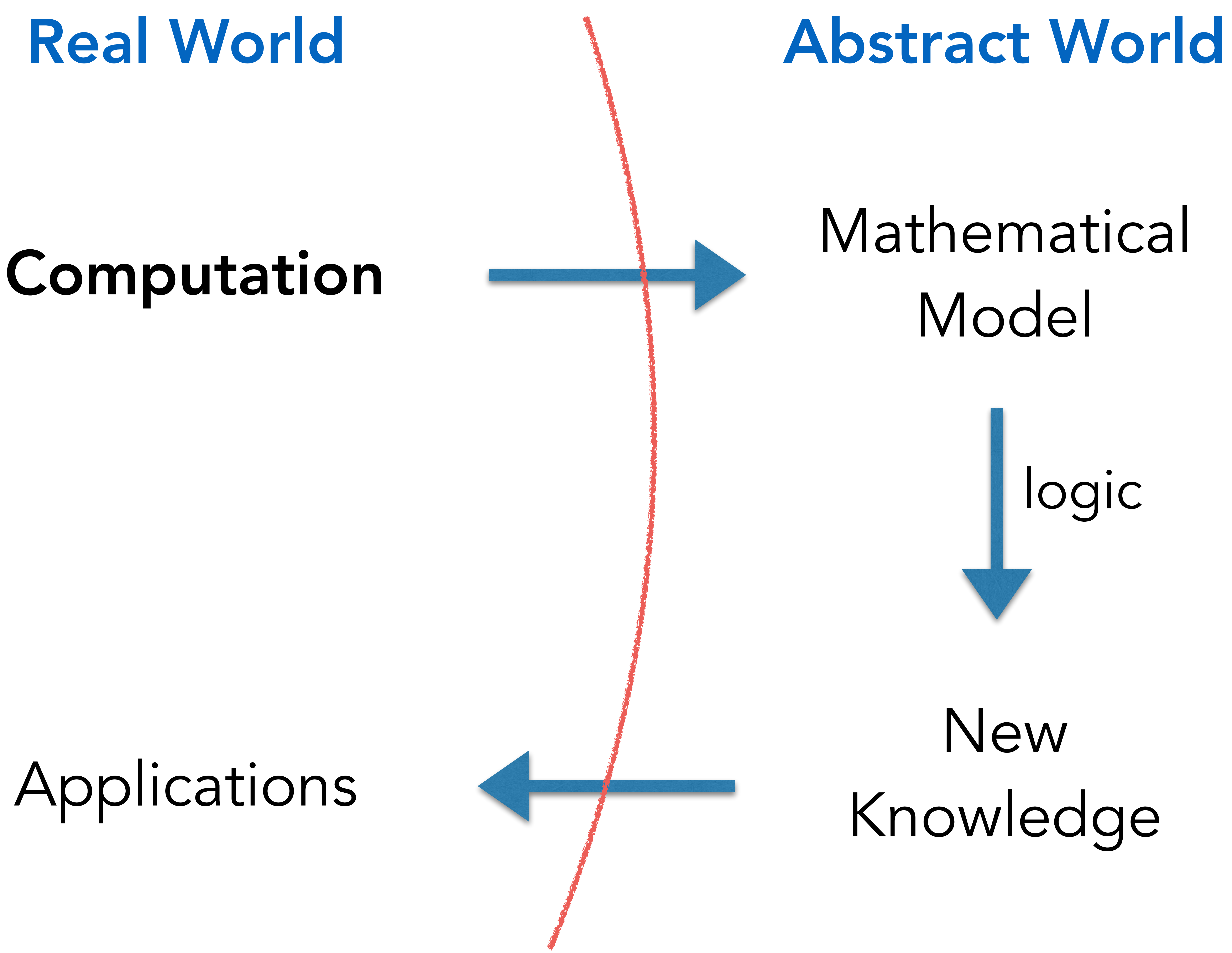
Next Few Chapters



What is **computation**?

What basic properties does it have?


Real World
Abstract World
Something of interest
Mathematical Model
logic
New Knowledge
Applications


Real World
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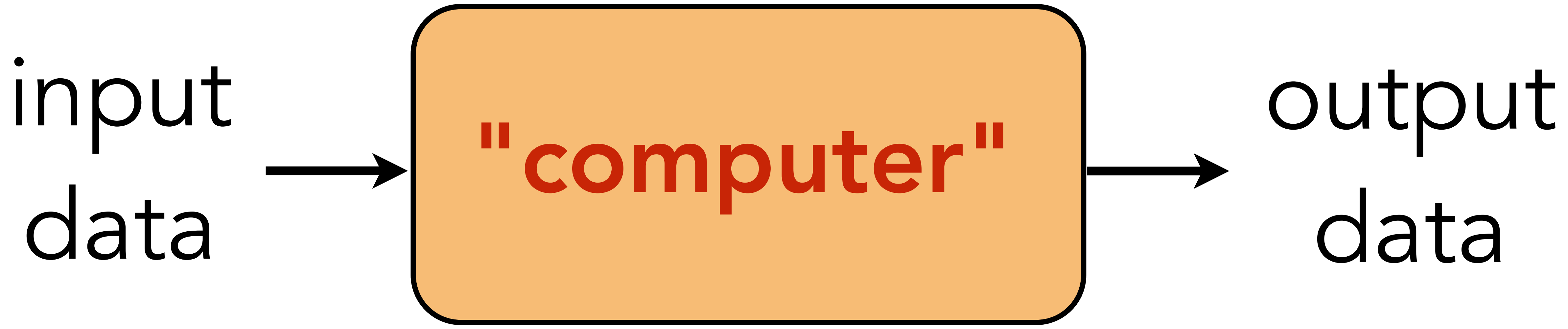
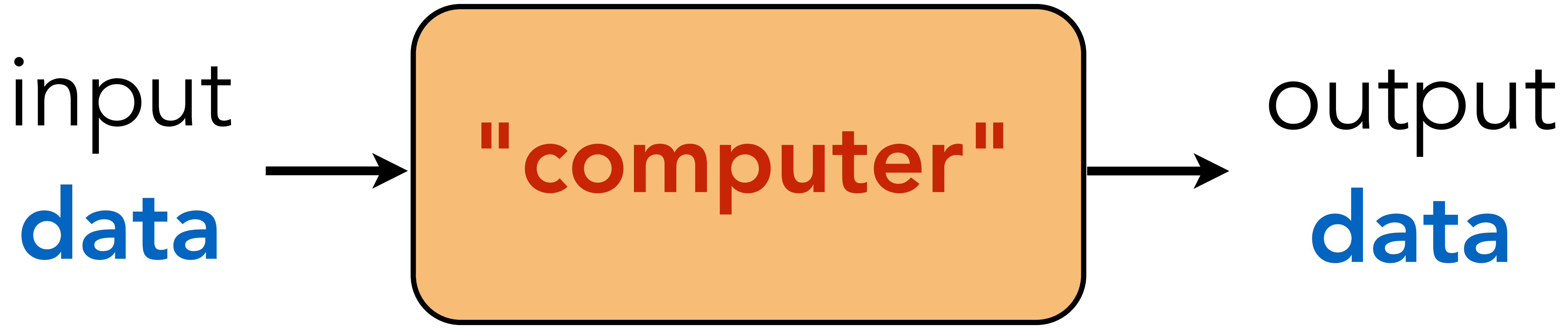
Now



Now



How do we mathematically represent **data**?

How we represent information

e.g. written communication:

"apple"

"car"

"happy"

"three" or "3"

English alphabet

$\Sigma = \{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w,x,y,z\}$

Turkish alphabet

Σ = {a,b,c,ç,d,e,f,g,ḡ,h,ı,i,j,k,l,m,n,o,ö,p,r,s,ş,t,u,ü,v,y,z}

?

What if we had more symbols?

What if we had less symbols?

Binary alphabet

$\Sigma = \{0, 1\}$

alphabet: a non-empty and finite set.
(usually denoted by Σ).

symbol/character: an element of an alphabet.

string/word: a finite sequence of symbols from Σ.

A string is denoted by $a_1a_2\ldots a_n$, where each $a_i \in \Sigma$.

(the definition sometimes includes infinite sequences)

Example: Some strings over $\Sigma = \{0,1\}$:

ϵ $\quad 0 \quad 1 \quad 01 \quad 101111010110111$

Example: Some strings over $\Sigma = \{a, b, c\}$:

$\epsilon \quad a \quad b \quad c \quad ca \quad caabcccab$

Length of a string s:

$$|s| = \text{the number of symbols in } s.$$

$\Sigma^* =$ set of all finite-length strings over Σ.

Examples:

$$\{0,1\}^* = \{\epsilon,0,1,00,01,10,11,000,001,0010,\ldots\}$$

$$\{a\}^* = \{\epsilon, a, aa, aaa, aaaa, \ldots\}$$

What is an encoding scheme?

$$\Sigma = \{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w,x,y,z\}$$

Objects/concepts		String encoding
	$\longrightarrow$	apple
	$\longrightarrow$	car
	$\longrightarrow$	happy

?

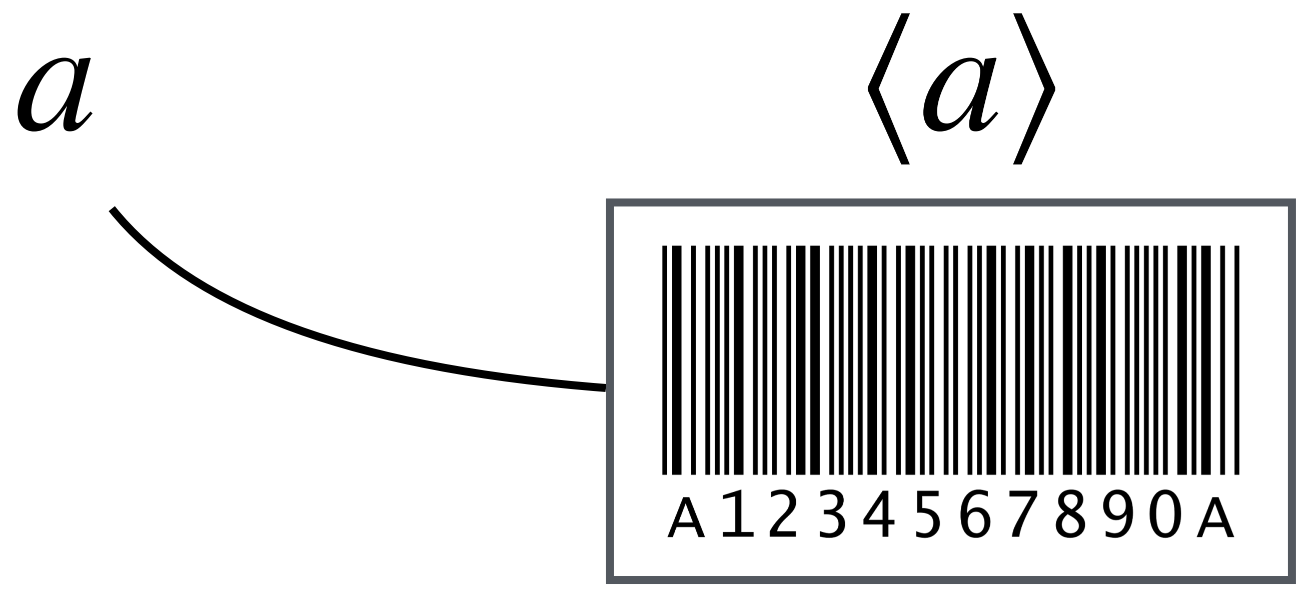
Does every object have a corresponding encoding?

Can two objects have the same encoding?

Does every string correspond to a valid encoding?

encoding: given a set A of objects, an encoding of elements of A is an **injective function** $\mathrm{Enc}\colon A \to \Sigma^*$.

For $a \in A$, $\langle a \rangle$ denotes $\mathrm{Enc}(a)$.



Warning: not all sets are encodable.

Examples $A = \mathbb{N}$

$\Sigma = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$\langle 36 \rangle =$ "36"

$\Sigma = \{0, 1\}$

$\langle 36 \rangle =$ "100100"

$\Sigma = \{1\}$

$\langle 36 \rangle =$ "111111111111111111111111111111111111"

? Does Σ affect encodability?

Examples $A = \mathbb{Z}$

$\Sigma = \{-, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$\langle -36 \rangle =$ "-36"

$\Sigma = \{0, 1\}$

$\langle -36 \rangle =$ "1100100"

$\Sigma = \{1\}$?

Examples $A = \mathbb{N} \times \mathbb{N}$

$$\Sigma = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, \#\}$$

$$\langle (3, 36) \rangle = \langle 3, 36 \rangle = \text{``3\#36''}$$

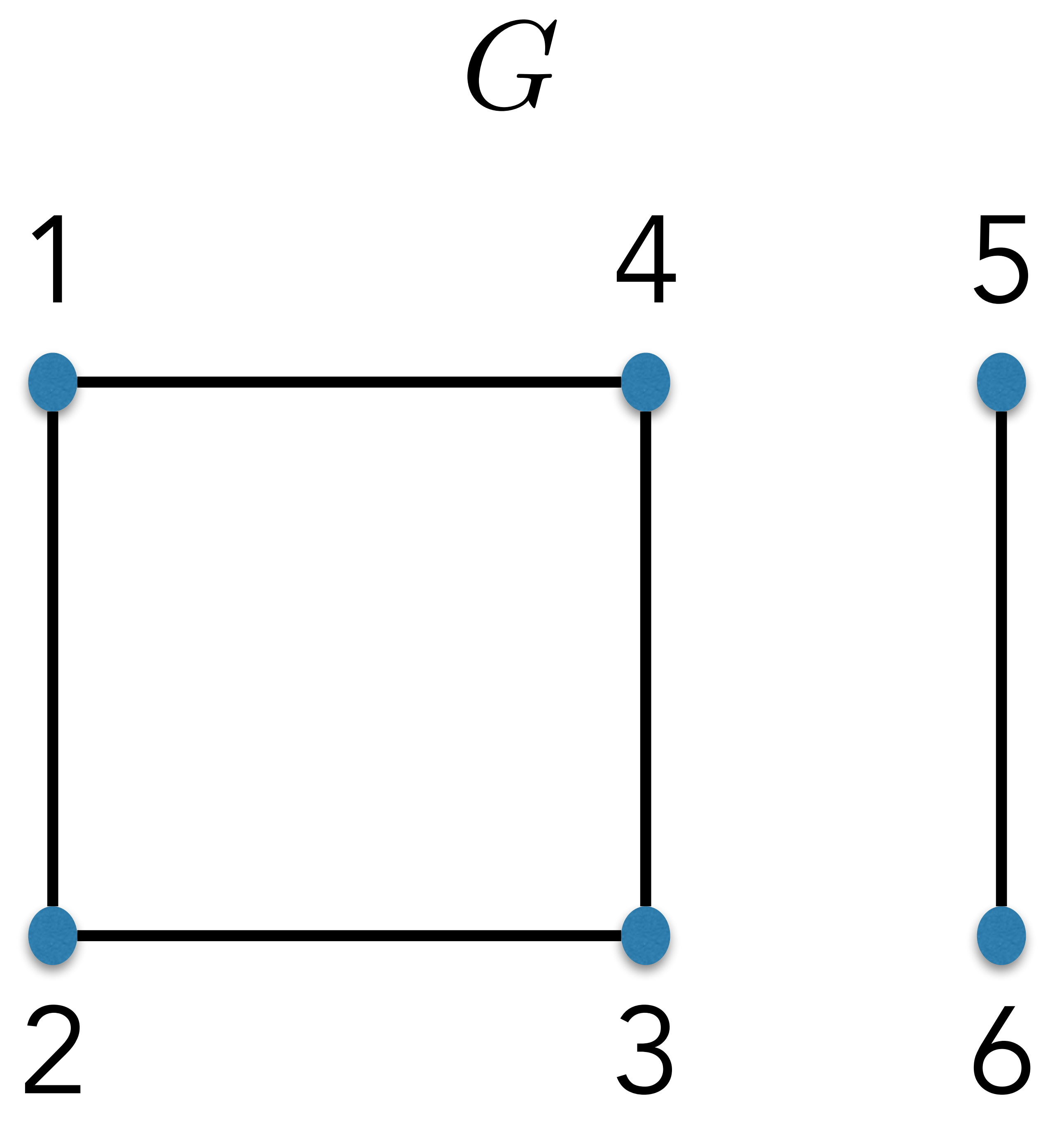
$$\Sigma = \{0, 1\}$$

Idea: encode all symbols above using 4 bits (why 4?)

$0 \rightarrow 0000$	$4 \rightarrow 0100$	$8 \rightarrow 1000$
$1 \rightarrow 0001$	$5 \rightarrow 0101$	$9 \rightarrow 1001$
$2 \rightarrow 0010$	$6 \rightarrow 0110$	$\# \rightarrow 1010$
$3 \rightarrow 0011$	$7 \rightarrow 0111$	

$$\langle 3, 36 \rangle = \text{``001110100011 0110''}$$

Examples $A =$ all undirected graphs



$\langle G \rangle =$ "V = {1, 2, 3, 4, 5, 6}
E = {{1,2}, {2,3}, {3,4}, {1,4}, {5,6}}"

Examples $A =$ all Python functions

```
def isPrime(N):
  if (N < 2):
    return False
  for factor in range(2, N):
    if (N % factor == 0):
      return False
  return True
```

$\langle \text{isPrime} \rangle =$

“def isPrime(N):\n if (N < 2):\n   return False\n for factor in range(2, N):\n   if (N % factor == 0):\n   return False\n return True”

Does $|\Sigma|$ matter?

Going from $|\Sigma| = k$ to $|\Sigma'| = 2$:

encode every symbol of Σ using t bits,

where $t = \lceil \log_2 k \rceil$.

A word of length n over Σ	→	A word of length tn over Σ'

Does $|\Sigma|$ matter?

$A = \mathbb{N}$ **Binary vs Unary**

	Binary	Unary
0	0	ϵ
1	1	1
2	10	11
3	11	111
4	100	1111
5	101	11111
6	110	111111
7	111	1111111
8	1000	11111111
9	1001	111111111
10	1010	1111111111
11	1011	11111111111
12	1100	111111111111

Does $|\Sigma|$ matter?

Binary vs Unary

n has length $\lfloor \log_2 n \rfloor + 1$ in **binary**

n has length $\lfloor \log_k n \rfloor + 1$ in **base** k

n has length n in **unary**

$$\log_k n = \frac{\log_2 n}{\log_2 k}$$

Unary is exponentially longer than other bases!

Which sets are encodable?

Encodability = Countability

(will see this later)

What about uncountable sets?

Approximate.

Summary So Far

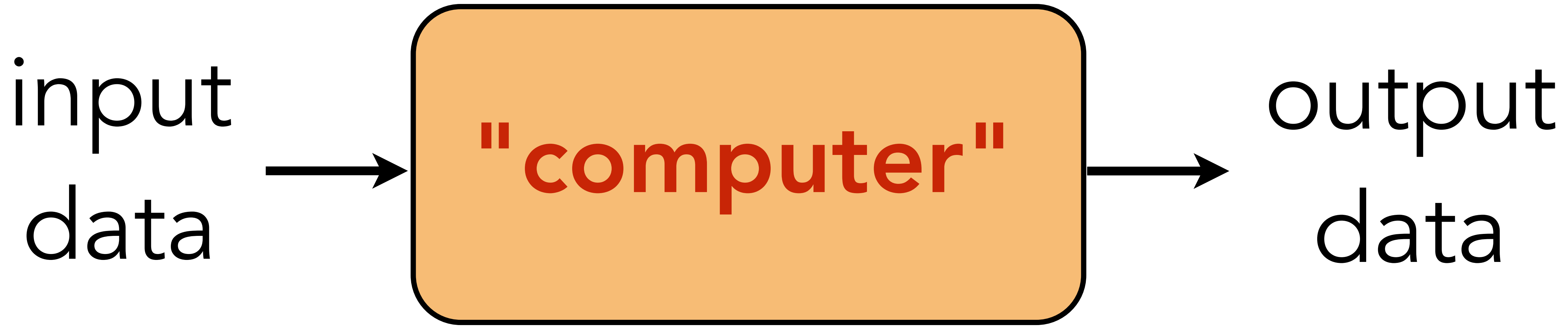
Alphabet Σ, String/word, Σ^*

Encoding of a set A: injective function $\text{Enc} : A \to \Sigma^*$.

Encodable = Countable

Alphabet doesn't matter much as long as $|\Sigma| > 1$.

Next Few Chapters

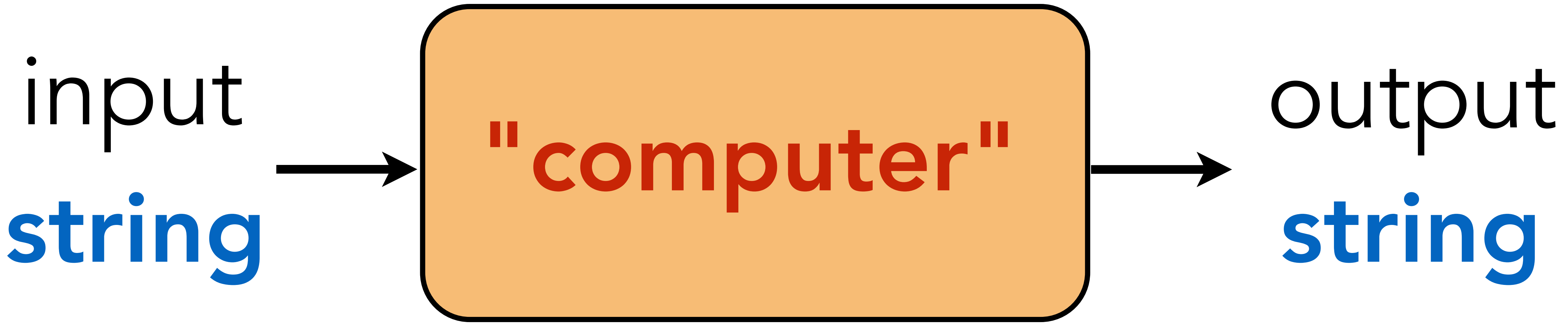


What is **computation**?

What is an **algorithm**?

How can we mathematically define them?

Next Few Chapters

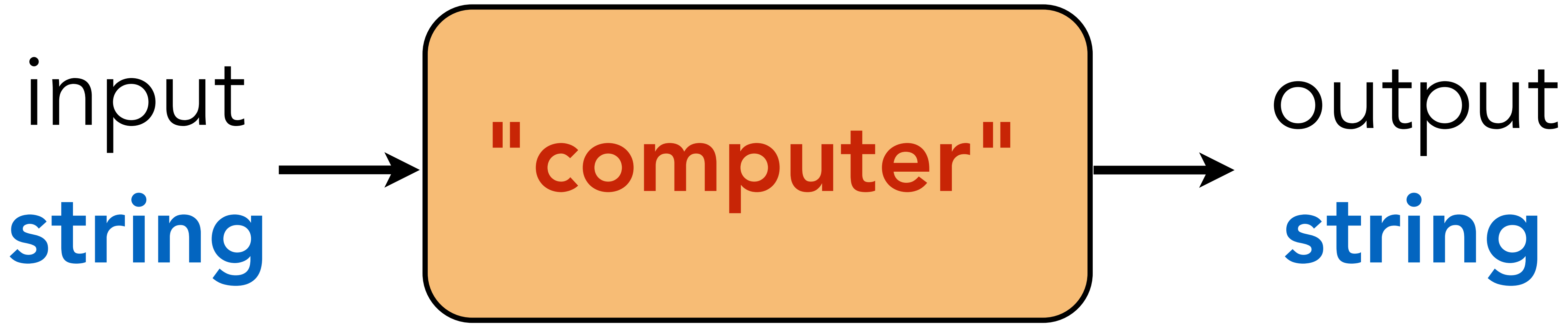


What is **computation**?

What is an **algorithm**?

How can we mathematically define them?

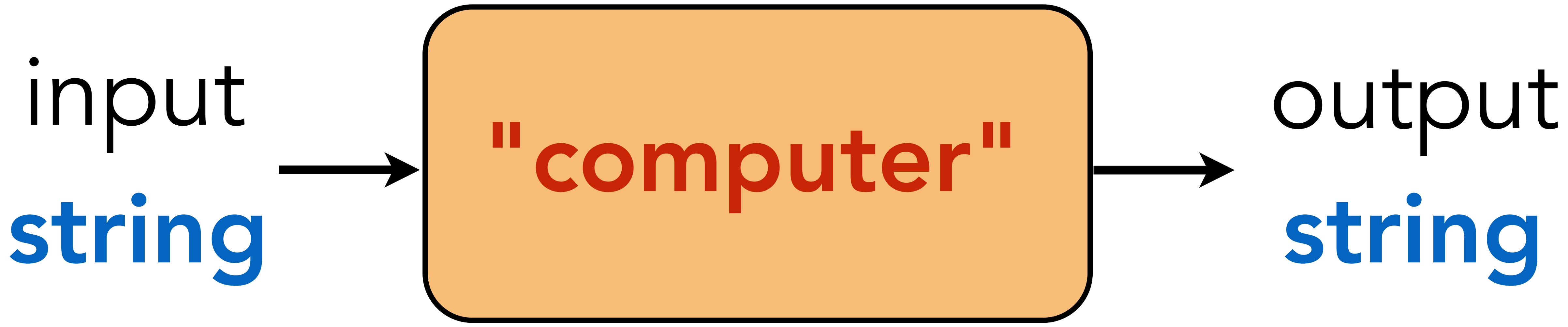
Let's lay the groundwork...



Reasonable assumptions to start with:

- Computer is *deterministic*
- Computation is a finite process
- Input can be any finite-length string
- For all inputs, there is an output
- Output is a finite-length string

? How can we characterize
the **input/output behavior** of a computer?



Is computer just a function $f : \Sigma^* \to \Sigma^*$ **???**

Function problem:
A function of the form $f : \Sigma^* \to \Sigma^*$.

A computer/algorithm **solves** function problem f
if its input/output behavior corresponds to f.

Function Problem Examples

$\Sigma = \{0,1\}$

Reverse function

$110100 \mapsto 001011$

Sort function

$110100 \mapsto 000111$

isPrime

$11111010 \mapsto 0$

$11111011 \mapsto 1$

Example description of a **function problem**:

Given a natural number N, output *True* if N is prime, and output *False* otherwise.

Input type: natural number

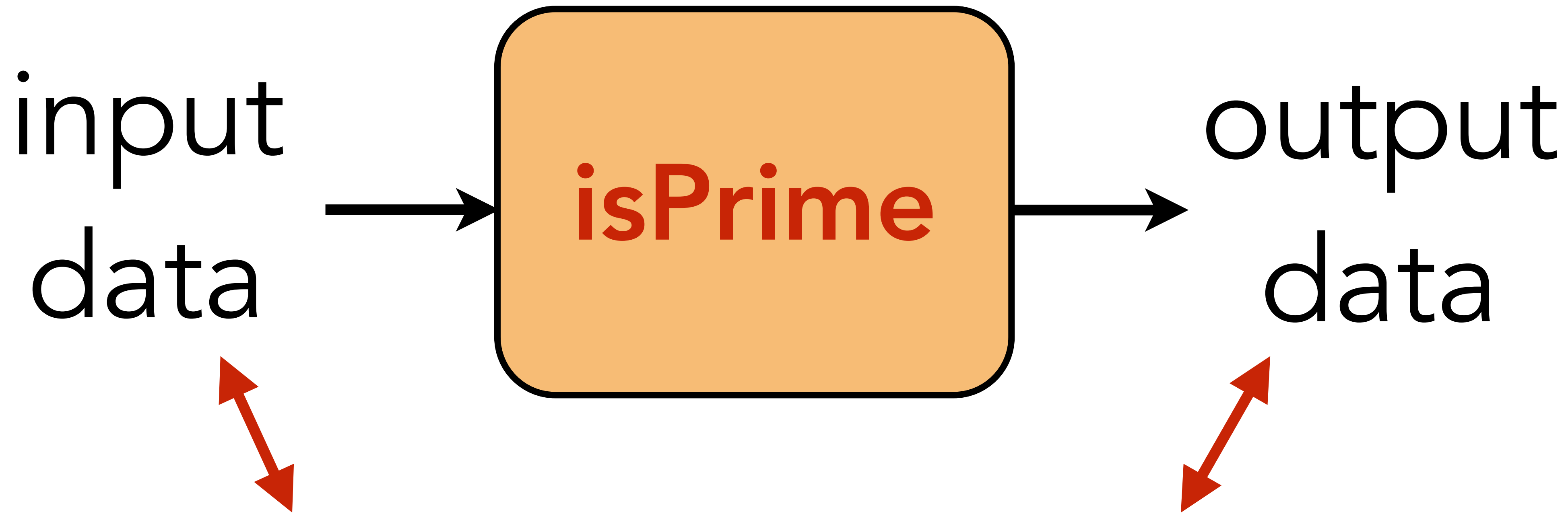
Output type: boolean

A **function problem** is a function

$$f : I \to S\ .$$

$I =$ set of possible input objects (called **instances**)

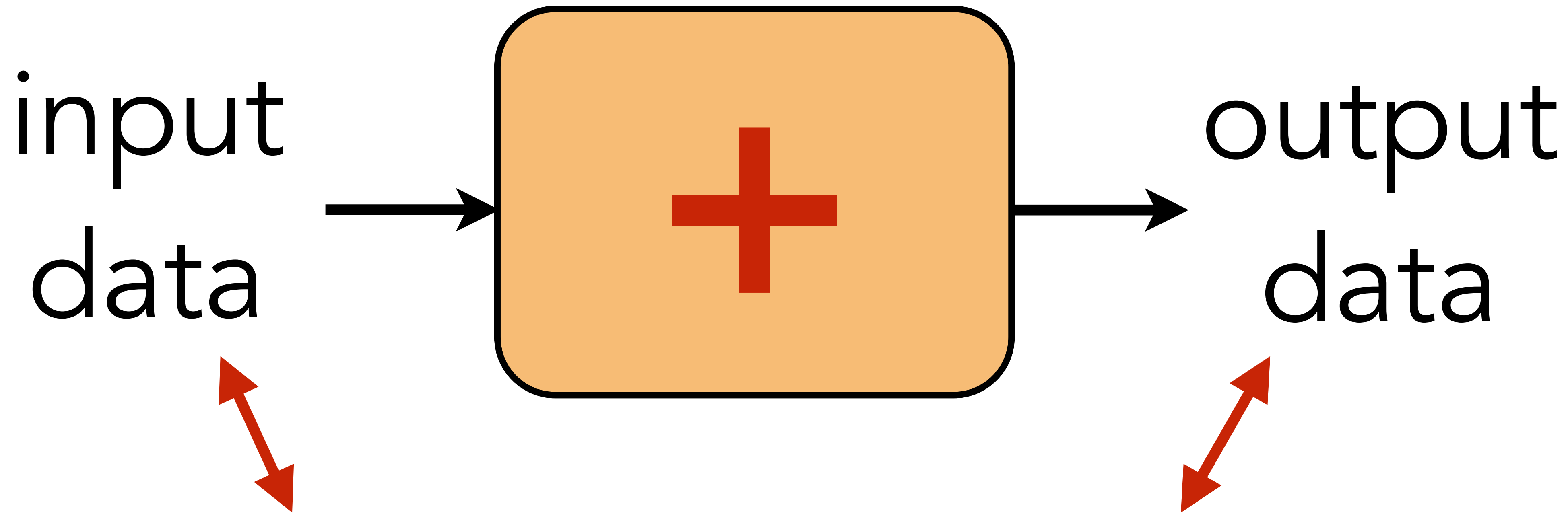
$S =$ set of possible output objects (called **solutions**)



$I = \mathbb{N}$

Instance	Solution
0	No
1	No
2	Yes
3	Yes
4	No
⋮	⋮
251	Yes
⋮	⋮

$S = \{\text{No}, \text{Yes}\}$



$I = \mathbb{N} \times \mathbb{N}$

Instance	Solution
0, 0	0
0, 1	1
1, 1	2
2, 2	4
2, 3	5
10, 1	11
100, 99	199
⋮	⋮

$S = \mathbb{N}$

Instance

["vanilla", "mind", "Anupam", "yogurt", "doesn't"]

Solution

["Anupam", "doesn't", "mind", "vanilla", "yogurt"]

A **function problem** is a function

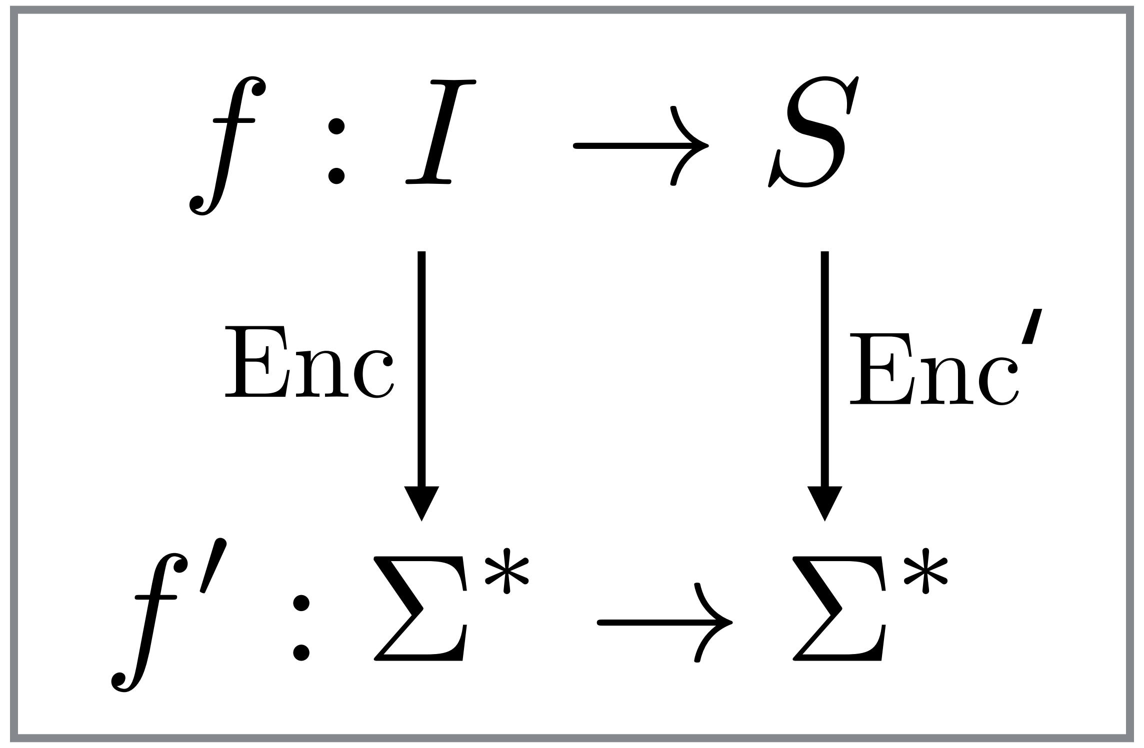
$$f : I \to S \,.$$

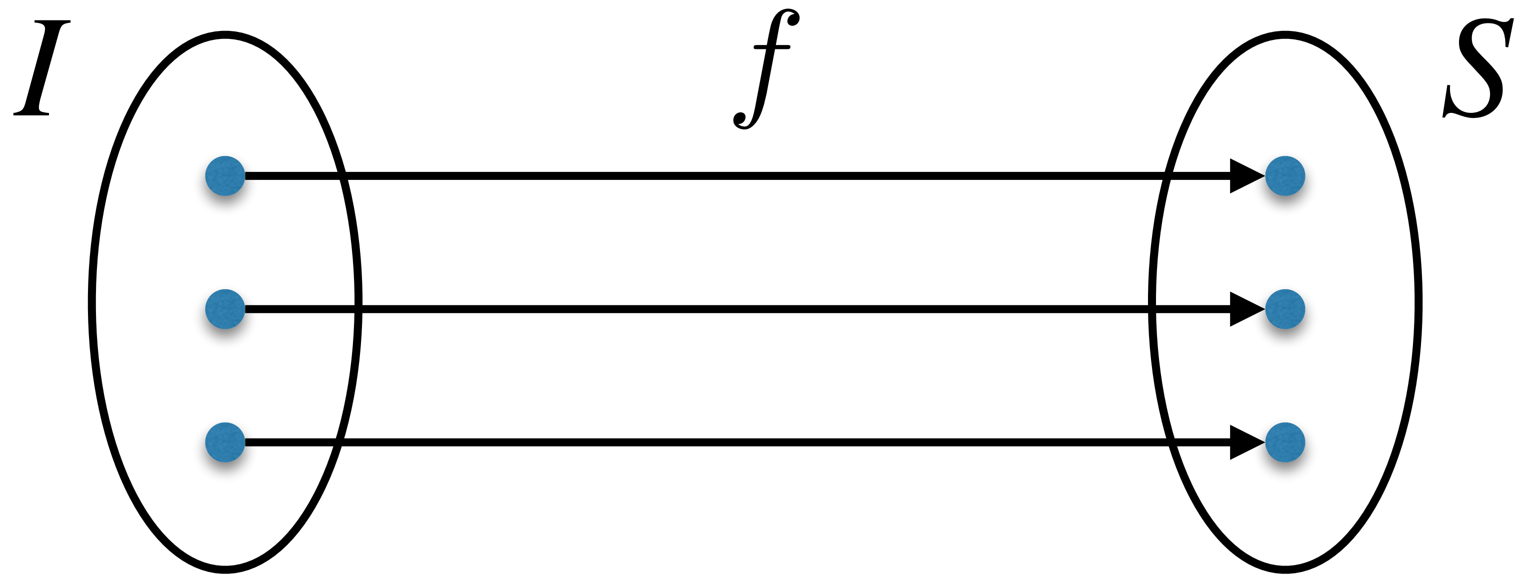
$I =$ set of possible input objects (called **instances**)

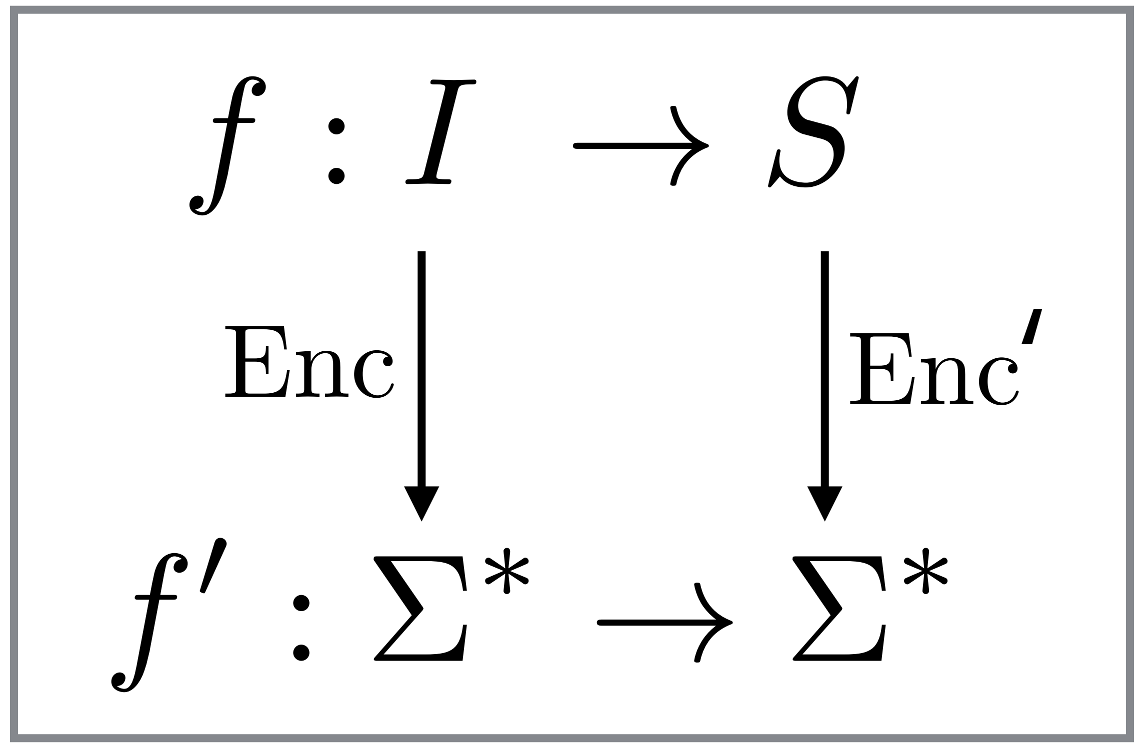
$S =$ set of possible output objects (called **solutions**)

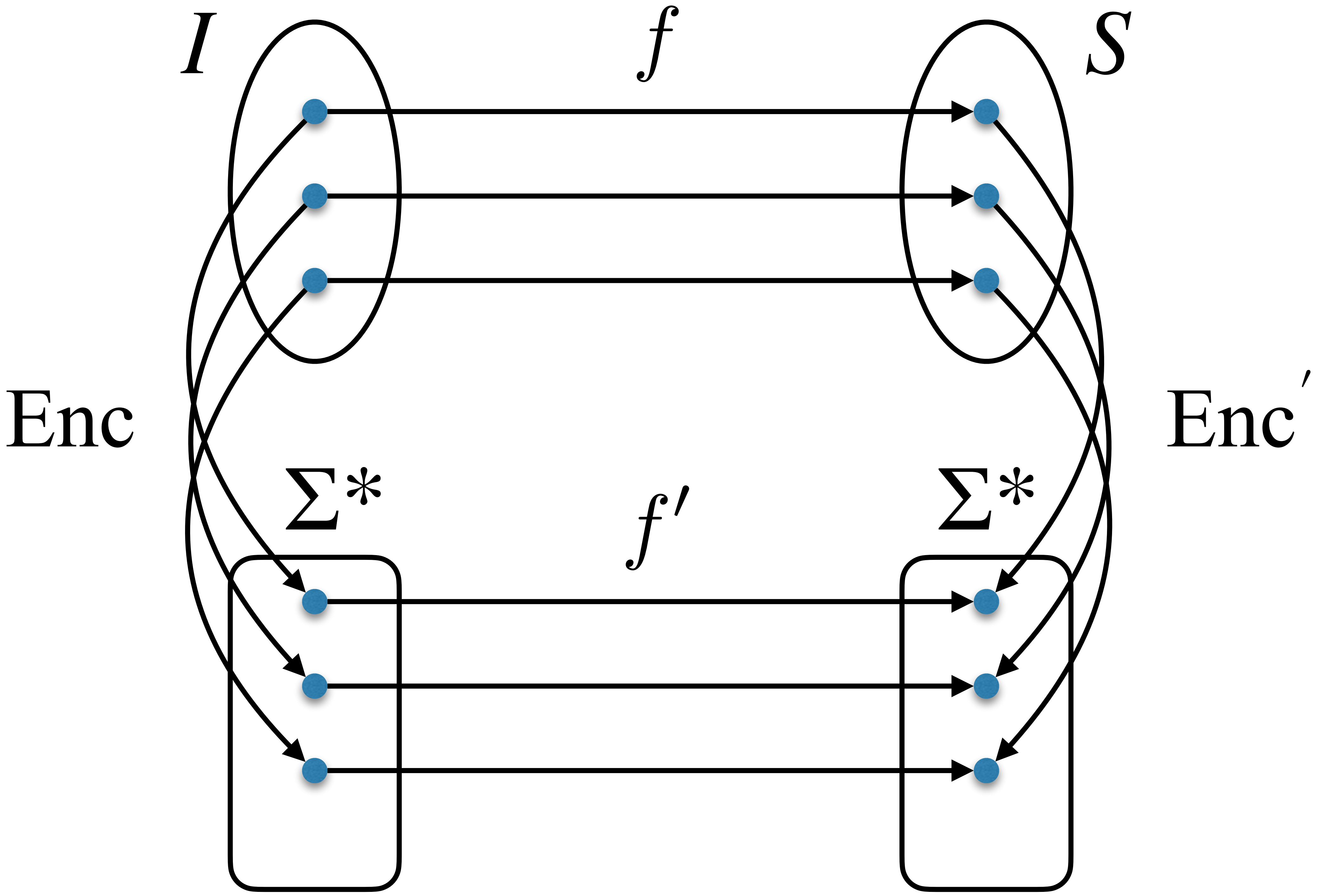
In TCS, we don't deal with arbitrary types,
we deal with **strings** (encodings).

$$\begin{array}{ccc} f : I & \to & S \\ \text{Enc}\downarrow & & \downarrow\text{Enc}' \\ f' : \Sigma^* & \to & \Sigma^* \end{array}$$


f : I → S
Enc
Enc′
f′ : Σ* → Σ*


I
f
S





Technicality: What if $w \in \Sigma^*$ does not correspond to an encoding of an instance?

In TCS, there is only one type of data:

string

A convenient restriction:

Problems with 2 possible solutions.

Example: isPrime

Decision problem:

A function of the form $f : \Sigma^* \rightarrow \{0,1\}$.

$\{\text{False, True}\}$

$\{\text{Reject, Accept}\}$

$\{\text{No, Yes}\}$

Why?

1. Simpler objects
2. Without loss of generality

1. Simpler objects

Language:

Any set L of finite-length strings over an alphabet Σ.

i.e. any set $L \subseteq \Sigma^*$.

Examples $\Sigma = \{0,1\}$

$L = \varnothing$

$L = \Sigma^*$

$L = \{0,\ 1,\ 00,\ 11\}$

$L = \{\epsilon,\ 01,\ 0011,\ 000111,\ \ldots\}$

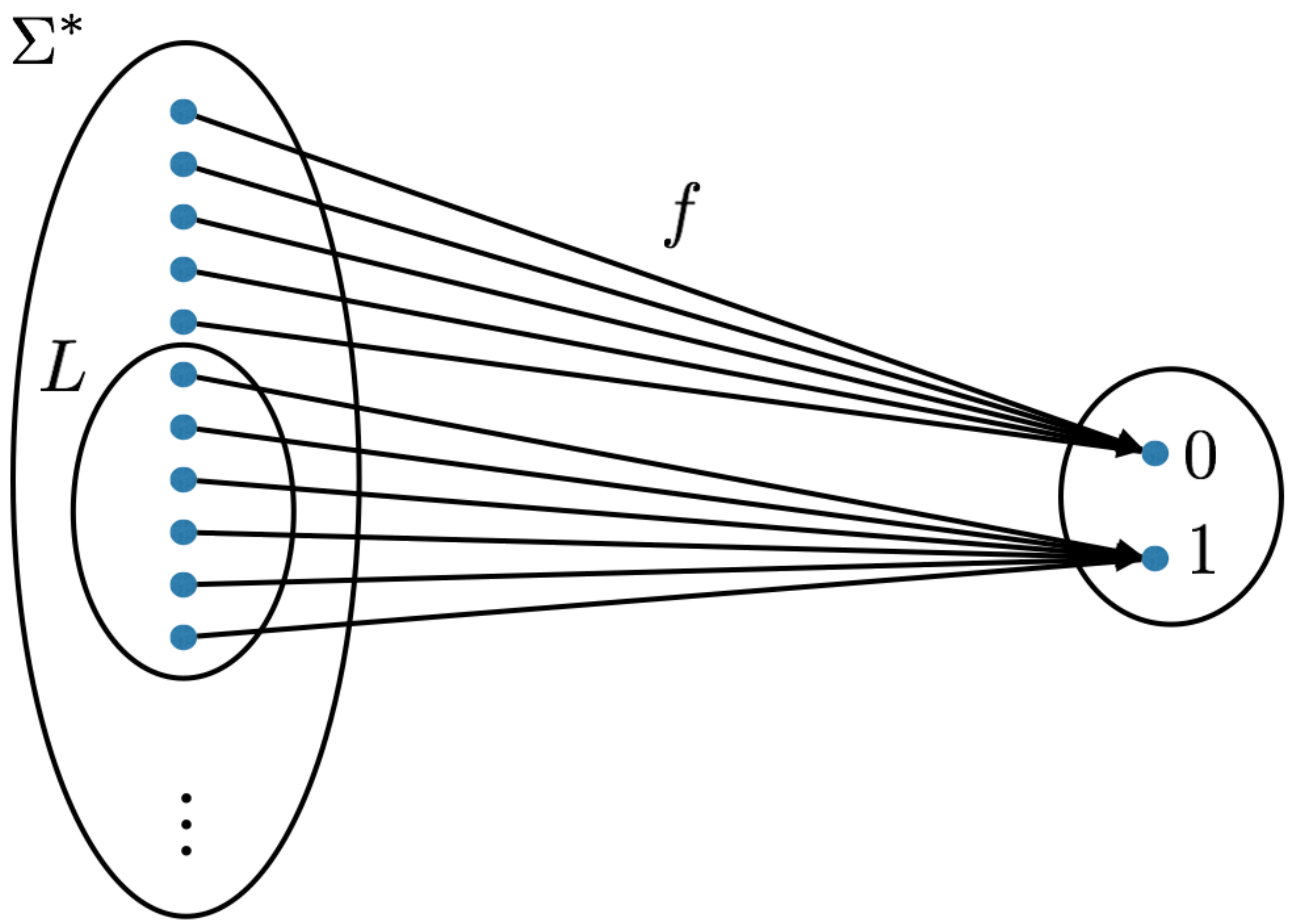
1. Simpler objects

There is a one-to-one correspondence between **decision problems** and **languages**.

$$f : \Sigma^* \rightarrow \{0,1\}$$

Instance	Solution
ϵ	1
0	1
1	1
00	1
01	0
10	0
11	1
000	1
001	0
$\vdots$	$\vdots$

$$L = \{\epsilon, 0, 1, 00, 11, 000, \ \ldots\}$$



2. Without loss of generality

function problem ≈ corresponding decision problem

Integer factorization problem:

Input: natural number N,

Output: prime factorization of N.

Decision version:

Input: natural numbers N and k,

Output: True iff N has a factor between 2 and k?

2. Without loss of generality

function problem $\approx$ corresponding decision problem

Integer factorization problem:

Input: natural number N,

Output: prime factorization of N.

Smallest factor problem:

Input: natural number N,

Output: smallest (prime) factor of N.

Decision version:

Input: natural numbers N and k,

Output: True iff N has a factor between 2 and k?

- Are all languages **computable/solvable**?
- How can you prove a language is not **solvable**?
- How do we measure the **complexity** of algorithms solving languages?
- How do we classify languages according to the resources needed to solve them?
- $P \stackrel{?}{=} NP$