Closure Properties of Regular Languages

Closure under complementation

Proposition: Let Σ be an alphabet. If $L \subseteq \Sigma^*$ is regular then so is $\overline{L} = \Sigma^* \setminus L$.

Proof:

Closure under union

Theorem: Let Σ be an alphabet. If $L_1 \subseteq \Sigma^*$ and $L_2 \subseteq \Sigma^*$ are regular then so is $L = L_1 \cup L_2$.

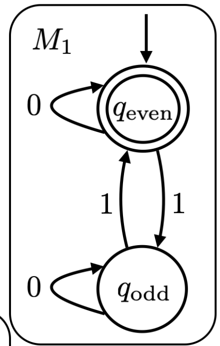
How does the proof start:

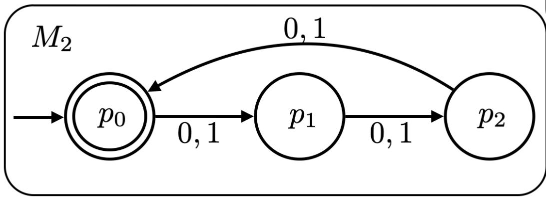
The mindset when constructing the DFA:

Example

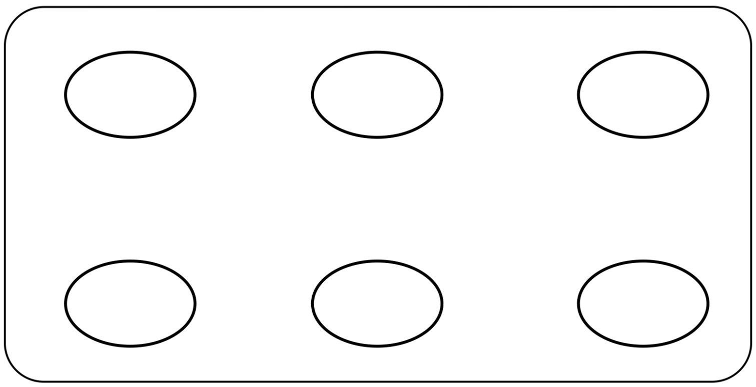
$L_1 =$ strings with even number of 1's

$L_2 =$ strings with length divisible by 3.





DFA for the union



Formally defining the DFA: Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA deciding L_1 and let $M' = (Q', \Sigma, \delta', q_0', F')$ be a DFA deciding L_2. We construct DFA $M'' = (Q'', \Sigma, \delta'', q_0'', F'')$ deciding $L_1 \cup L_2$ as follows.

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Simple vs Regular

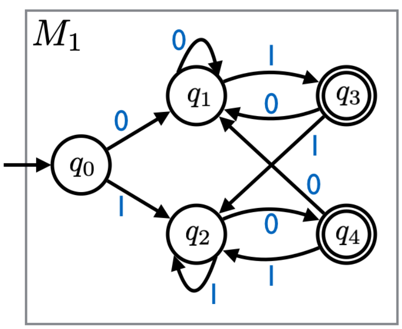
Theorem: Can define regular languages recursively as follows:

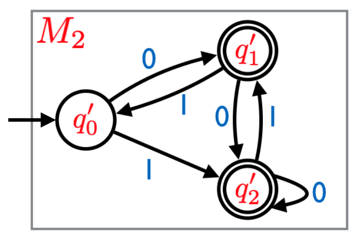
- $\emptyset$ is regular;
- For every $a \in \Sigma$, $\{a\}$ is regular;
- L_1, L_2 regular $\implies$ $L_1 \cup L_2$ regular;
- L_1, L_2 regular $\implies$ $L_1 L_2$ regular;
- L regular $\implies$ L^* regular.

Closure under concatenation

Theorem: Let Σ be an alphabet. If $L_1 \subseteq \Sigma^*$ and $L_2 \subseteq \Sigma^*$ are regular then so is $L = L_1 L_2$.

Example:





Given $w \in \Sigma^*$, we need to decide if

$$w = uv \quad \text{for} \quad u \in L_1, v \in L_2.$$

Problem: Don't know where u ends and where v begins.
When do you stop simulating M_1 and start simulating M_2?

Suppose you know u ends at w_3.

	0	0	1	1	0	0	1	0	0	1
	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	w_9	w_{10}
q_0	q_1	q_1	(q_3)							
			q'_0	q'_2	q'_2	q'_2	q'_1	q'_2	q'_2	q'_1

Thread:

Keep track of all threads:

		0	0	1	1	0	0	1	0	0	1	1
		w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	w_9	w_{10}	w_{11}
	q_0	q_1	q_1	(q_3)	q_2	(q_4)	q_1	(q_3)	q_1	q_1	(q_3)	q_2
thread1				q_0'	q_2'	q_2'	q_2'	q_1'	q_2'	q_2'	q_1'	q_0'
thread2						q_0'	q_1'	q_0'	q_1'	q_2'	q_1'	q_0'
thread3								q_0'	q_1'	q_2'	q_1'	q_0'
thread4											q_0'	q_2'

Problem:

Solution:

Formally defining the DFA: Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA deciding L_1 and let $M' = (Q', \Sigma, \delta', q_0', F')$ be a DFA deciding L_2. We construct DFA $M'' = (Q'', \Sigma, \delta'', q_0'', F'')$ deciding $L_1 L_2$ as follows.

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