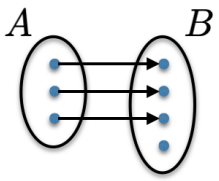
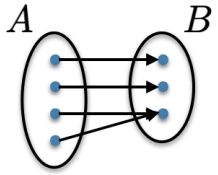
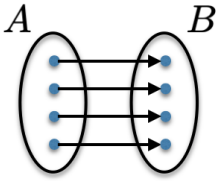
**Galileo on comparing cardinalities of infinite sets:**

**Georg Cantor (1845-1918)** **2 Great Ideas:**

**3 Important types of functions**




**Definition of $|A| < |B|$:**

**Sanity checks**

$|A| \leq |B|$ if and only if $|B| \geq |A|$.

$|A| = |B|$ if and only if $|A| \leq |B|$ and $|A| \geq |B|$.

If $|A| \leq |B|$ and $|B| \leq |C|$, then $|A| \leq |C|$.

## Examples of bijections

$|\mathbb{N}| = |\mathbb{Z}|$

**Heuristic for showing $|A| = |\mathbb{N}|$:**

$|\mathbb{N}| = |\text{Squares}|$

$|\mathbb{N}| = |\text{Primes}|$

$|\mathbb{N}| = |\mathbb{Z} \times \mathbb{Z}|$

$|\mathbb{N}| = |\{0,1\}^*|$

$|\mathbb{N}| = |\Sigma^*|$

**The Picture:**

**2 Definitions of Countable Set:**

Is $\mathbb{Q}$ countable?

Is $\mathbb{Q}[x]$ countable?

## Diagonalization

| $\mathscr{F}$ Functions | Inputs $X$ | | | |
|---|---|---|---|---|
| | $x_1$ | $x_2$ | $x_3$ | $x_4$ |
| $f_1$ | 0 | 0 | 1 | 0 |
| $f_2$ | 1 | 1 | 1 | 0 |
| $f_3$ | 1 | 0 | 0 | 0 |
| $f_4$ | 1 | 0 | 1 | 1 |
| $f_D$ | | | | |

Given:

Goal:

How:

Condition needed:

**Diagonalization Lemma:**

Picture:

**Corollary:**

**Diagonalization Lemma (General Case):**

**Definition:**

**Cantor's Theorem:**

**Corollary 1:**

**Corollary 2:**

**Important Note:** Diagonalization produces an **explicit** $f_D$ outside of $\mathcal{F}$.

## Limits of Computation

Picture: