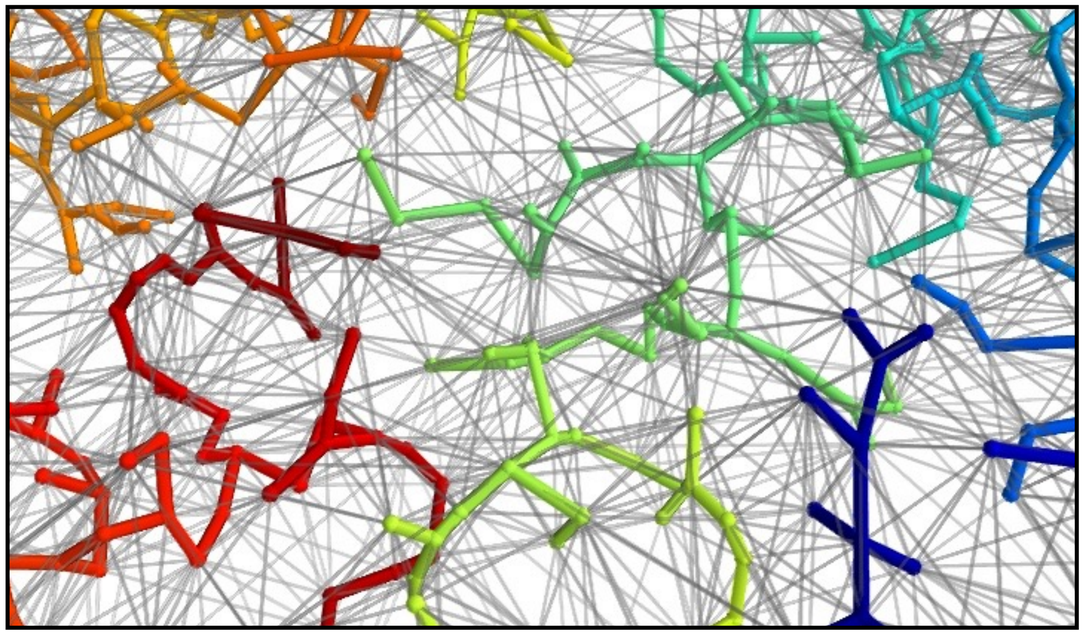
CS251

Great Ideas in *Theoretical* Computer Science

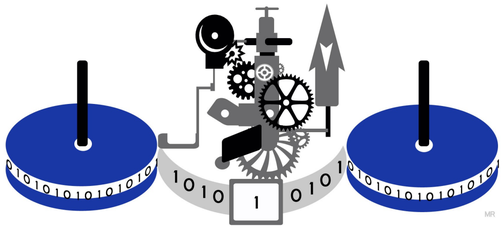


Maximum Matchings

Some real-world examples

matching machines and jobs

each machine is capable of doing a subset of the jobs



Job 1

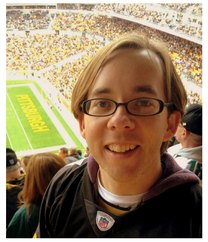
Job 2

⋮

Job n

Some real-world examples

matching professors and courses



⋮

15-110

15-112

15-122

15-150

15-251

15-259

⋮

Some real-world examples

matching rooms and courses

GHC 4401	15-110
DH 2210	15-112
GHC 5222	15-122
WEH 7500	15-150
DH 2315	15-251
⋮	⋮

Some real-world examples

matching students and internships

Macrosoft

Moogle

UMBRELLA CORPORATION

"Our business is life itself..."

How do you solve a problem like this?

1. Formulate the problem

2. **Ask**: Is there a trivial algorithm? Find and analyze.

3. **Ask**: Is there a better algorithm? Find and analyze.

Step 1: Formulate the problem

Purpose:

- Get rid of distractions, identify the crux.

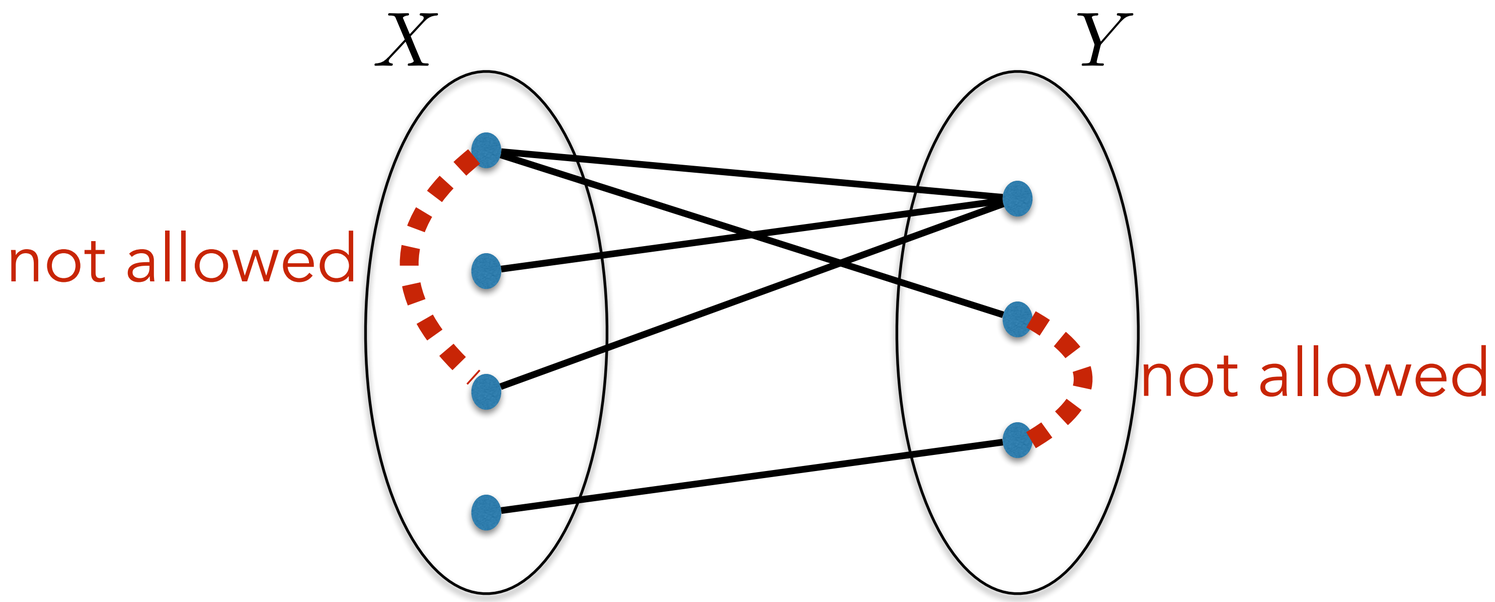
- Get a clean mathematical model (easier to reason about).

- Solutions often generalize to other settings.

CS Life Lesson

If your problem has a graph, great!!!
If not, try to make it have a graph.

Bipartite Graphs

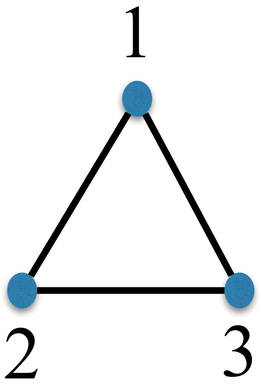


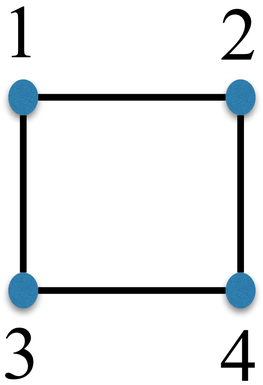
$G = (V, E)$ is **bipartite** if:

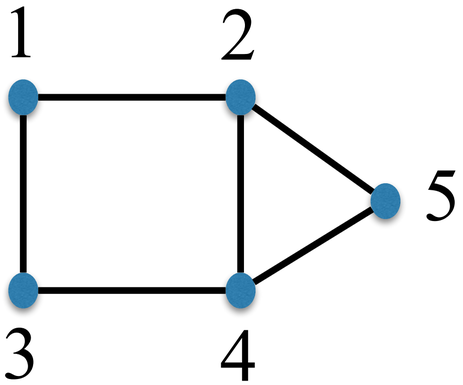
- there exists a bipartition of V into X and Y,
- each edge connects a vertex in X to a vertex in Y.

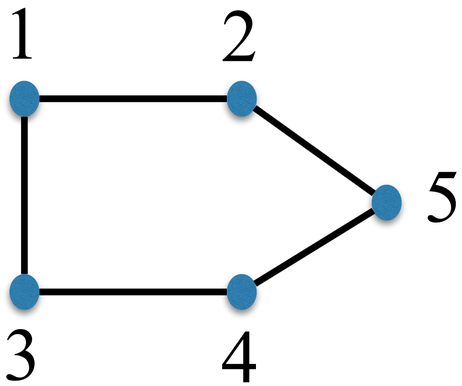
Bipartite Graphs

Given a graph $G = (V, E)$, we can ask, is it bipartite?



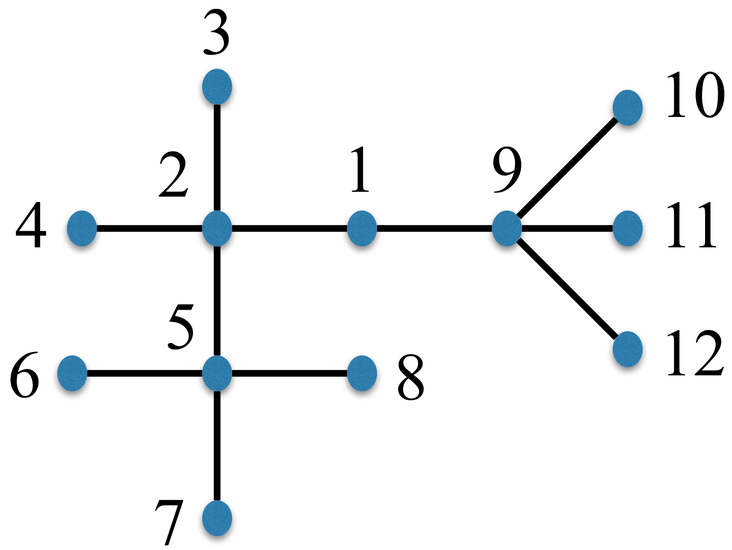




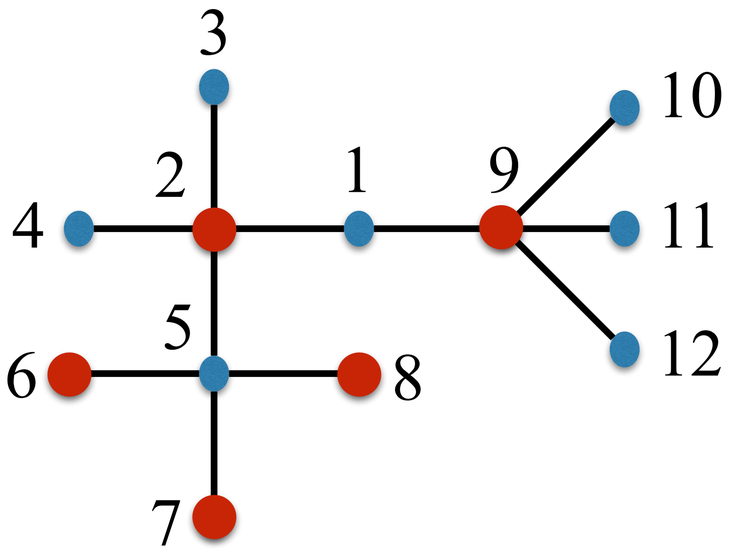


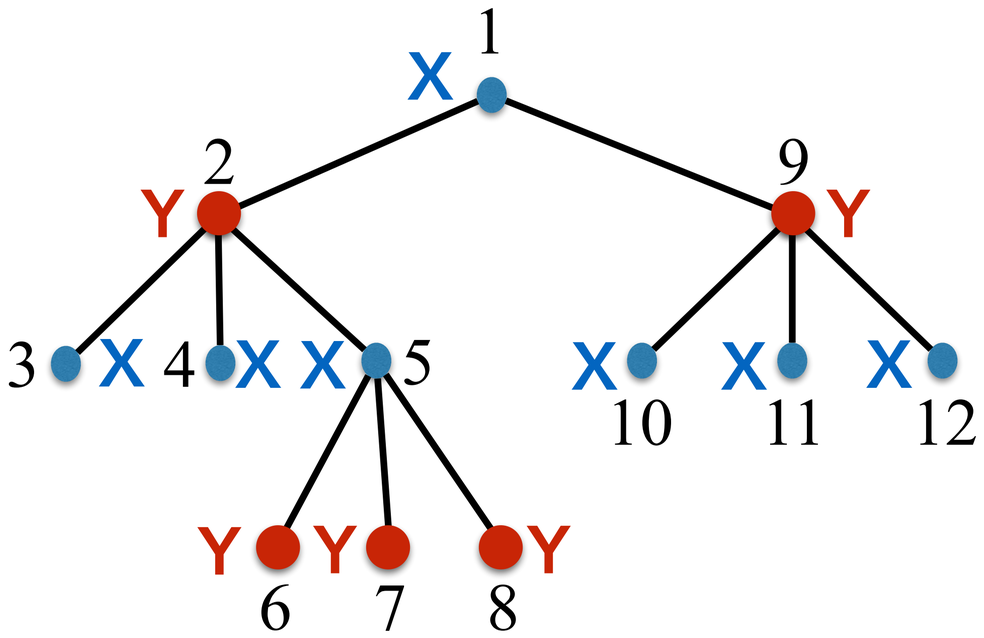
poll.cs251.com

Is this bipartite?



Poll Answer





bipartite = 2-colorable

k-colorable graph: Can color the vertices with **k colors** so that no edge's endpoints get the same color.

Characterization of Bipartite Graphs

An obstruction for being bipartite:

Contains a cycle of odd length.

Is this the only type of obstruction?

Theorem: A graph $G = (V, E)$ is bipartite
if and only if it contains no cycles of odd length.

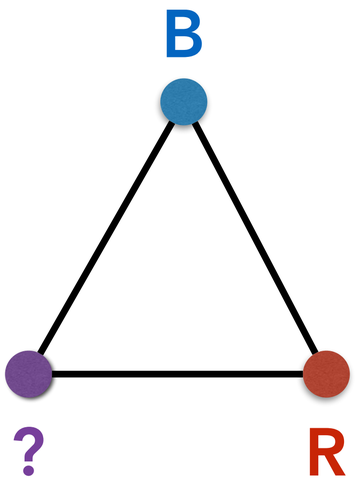
Proof

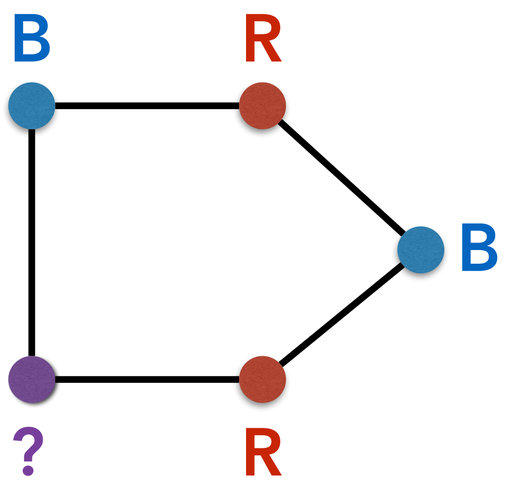
G is bipartite **iff** G contains no odd-length cycles.

Proof

G is 2-colorable **iff** G contains no odd-length cycles.

If G has an odd-length cycle, then it is not 2-colorable.





Proof

G is 2-colorable **iff** G contains no odd-length cycles.

If G has no odd-length cycle, then it is 2-colorable.

Strategy:

- Present a proper 2-coloring of the graph.
- Can assume the graph is connected.
  (we can 2-color each connected component separately)

Proof

G is 2-colorable **iff** G contains no odd-length cycles.

If G has no odd-length cycle, then it is 2-colorable.

Distance from root

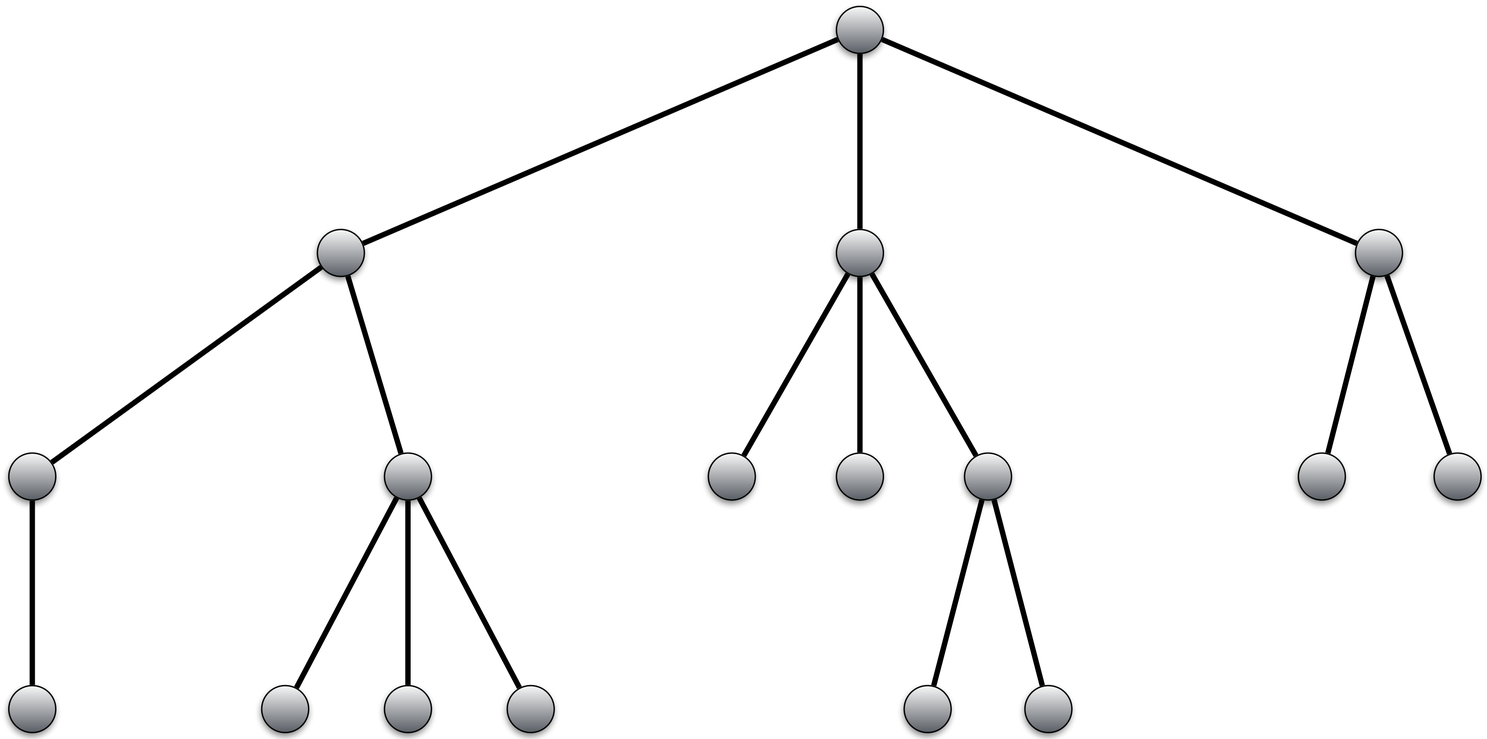
0

1

2

3

BFS tree



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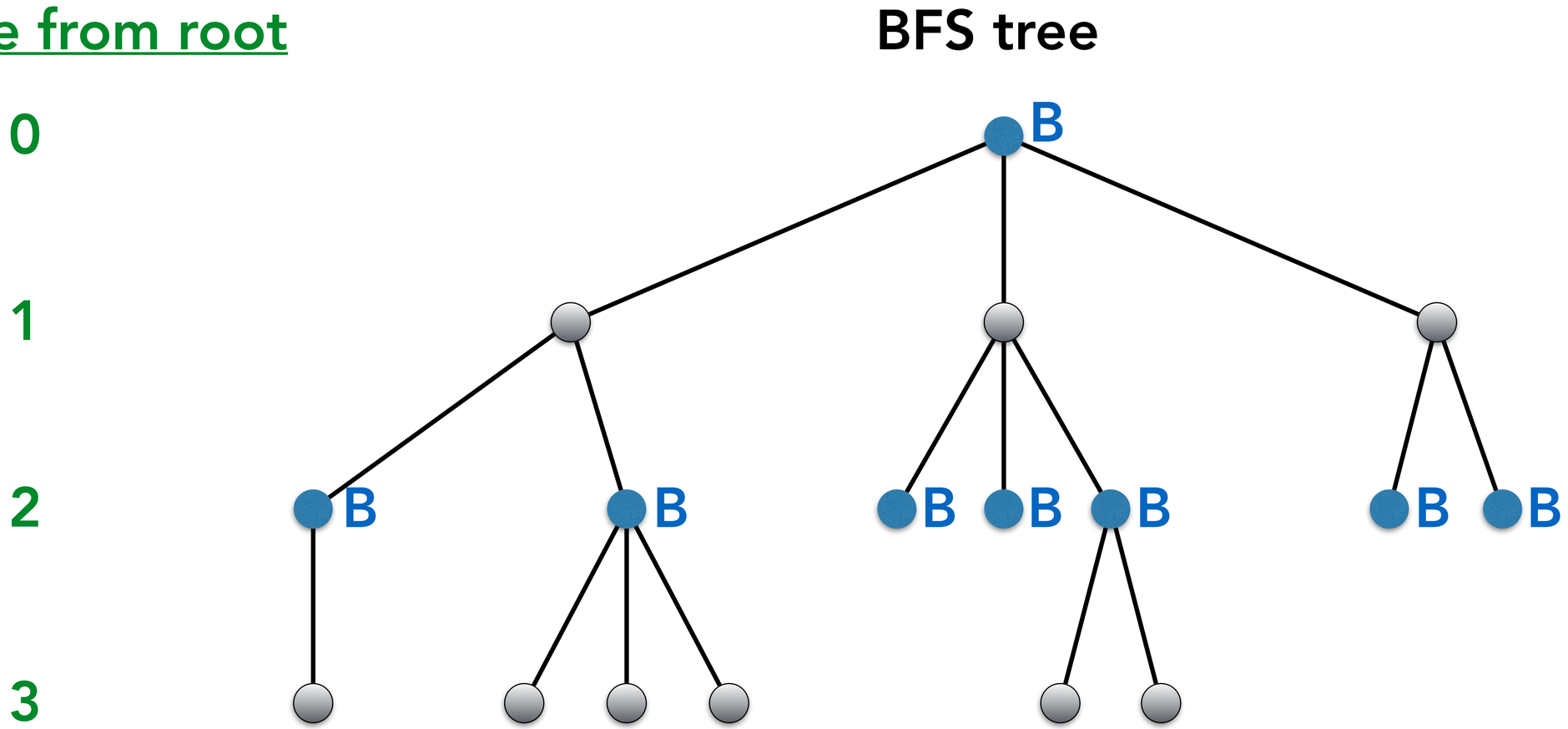

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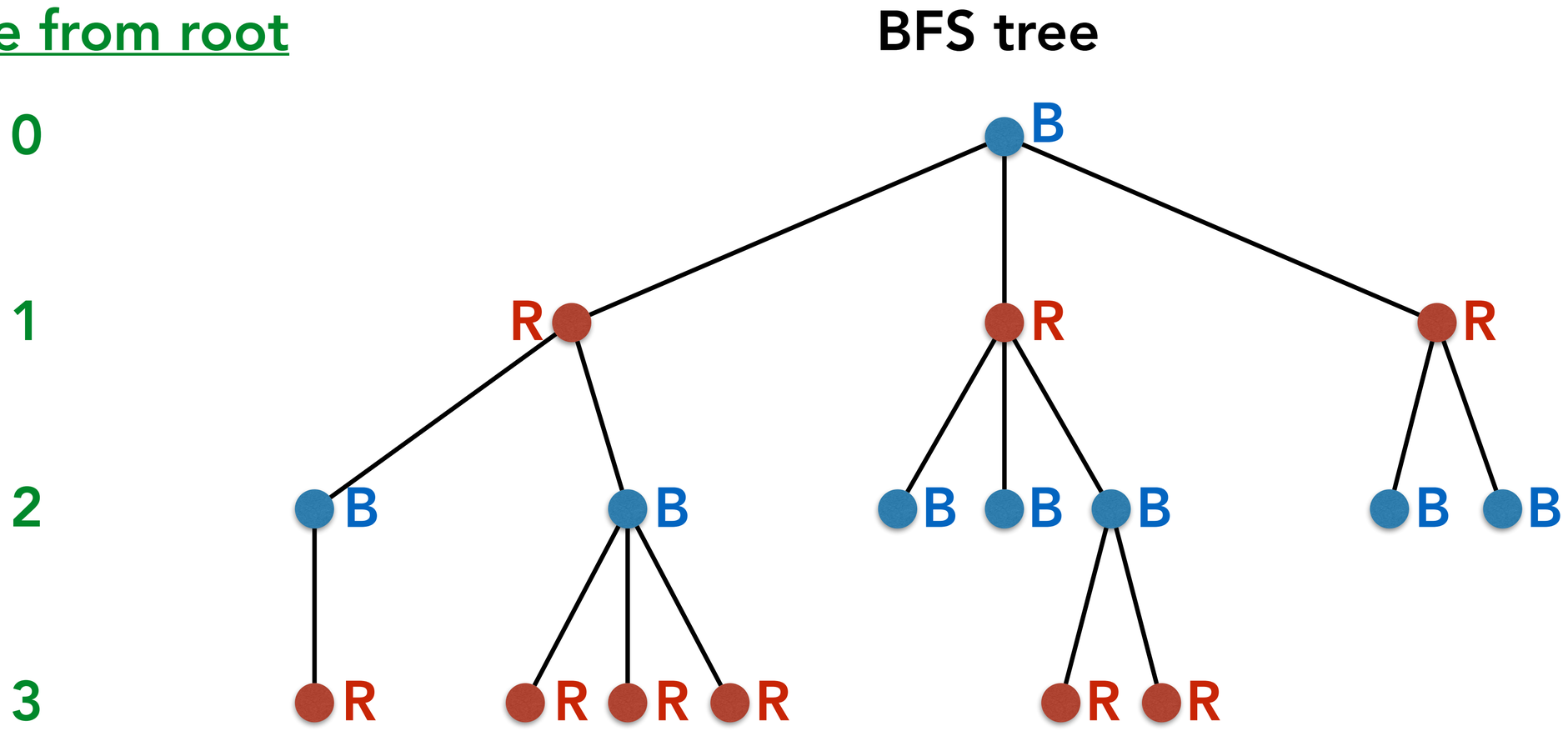

Can there be an edge whose endpoints are colored the same?

Proof

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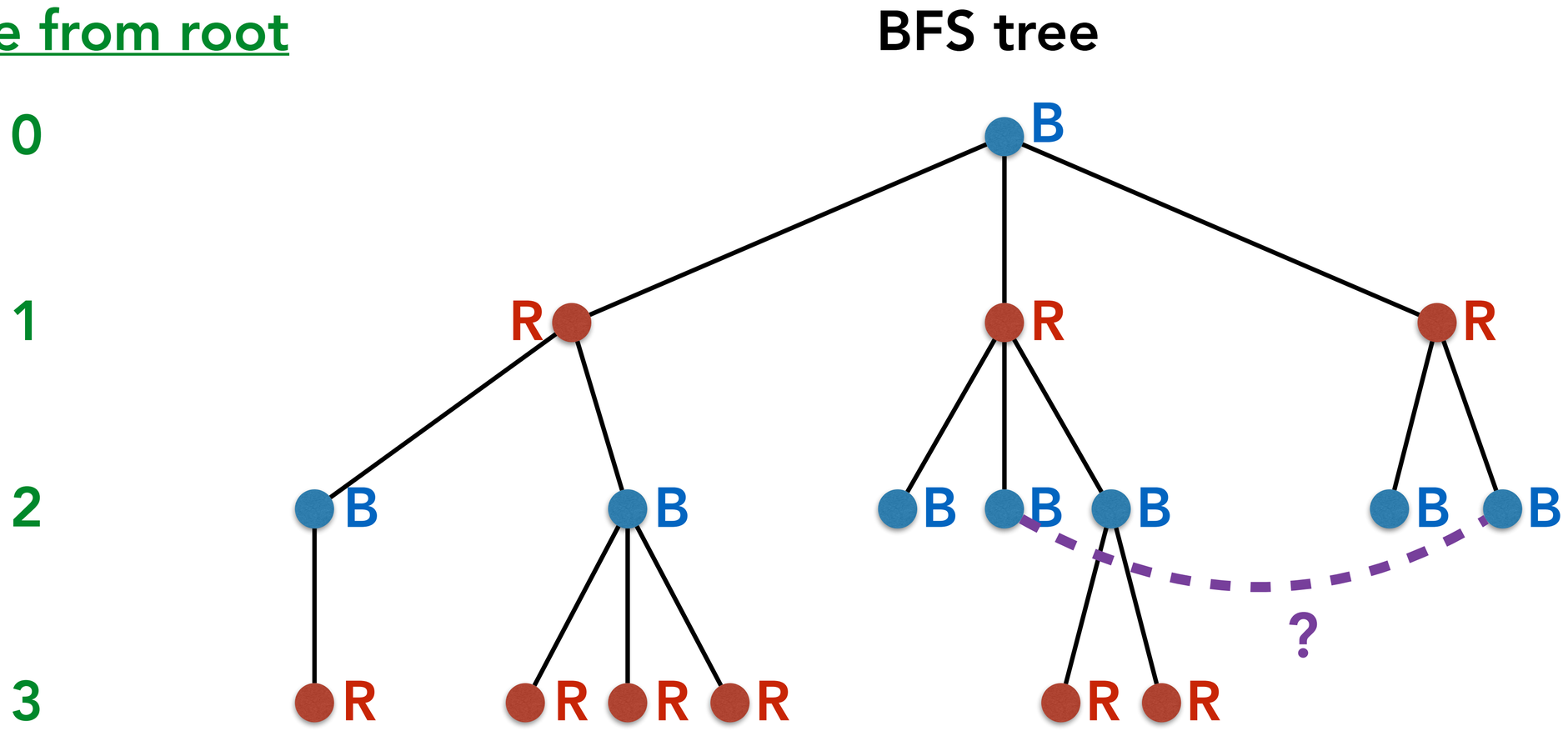

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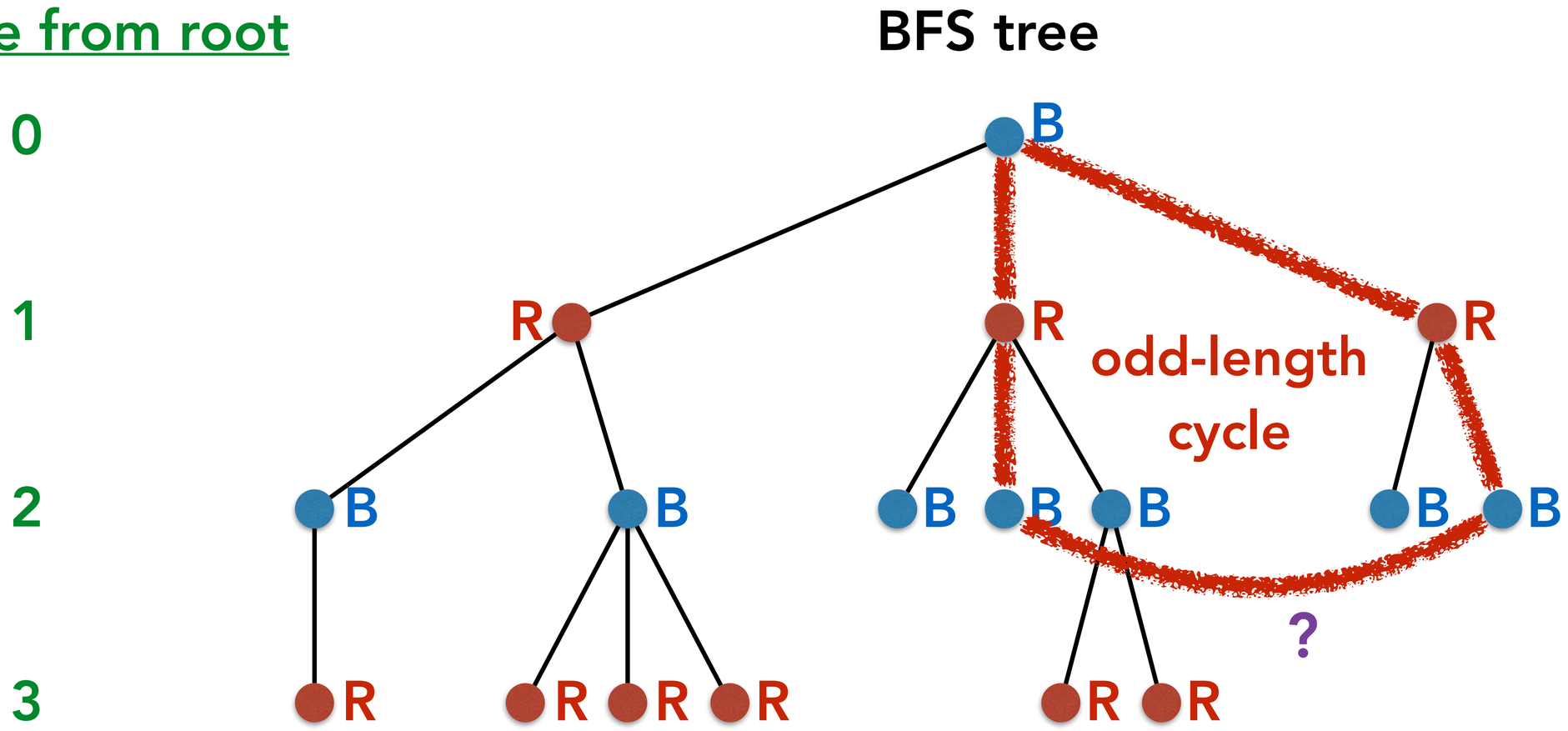

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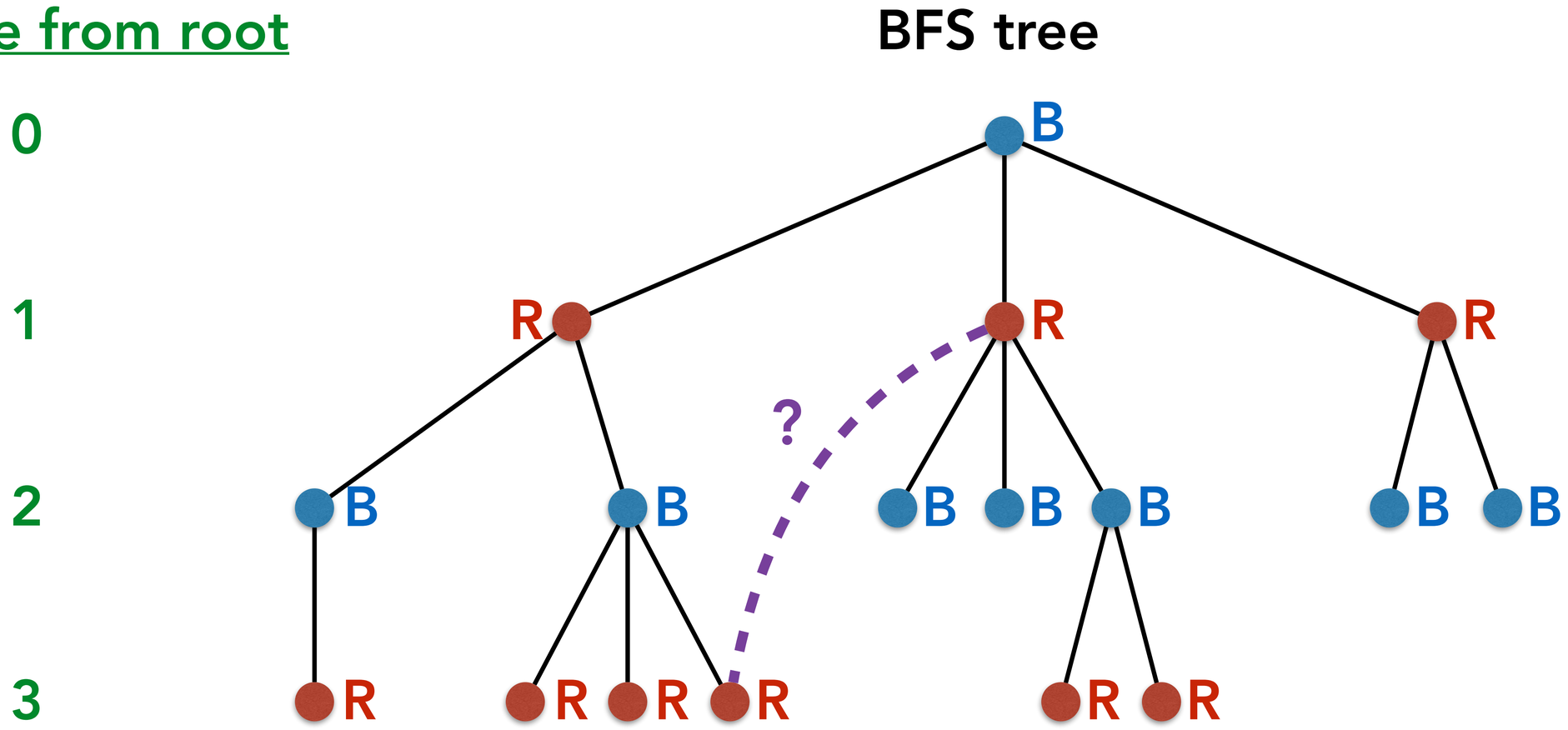

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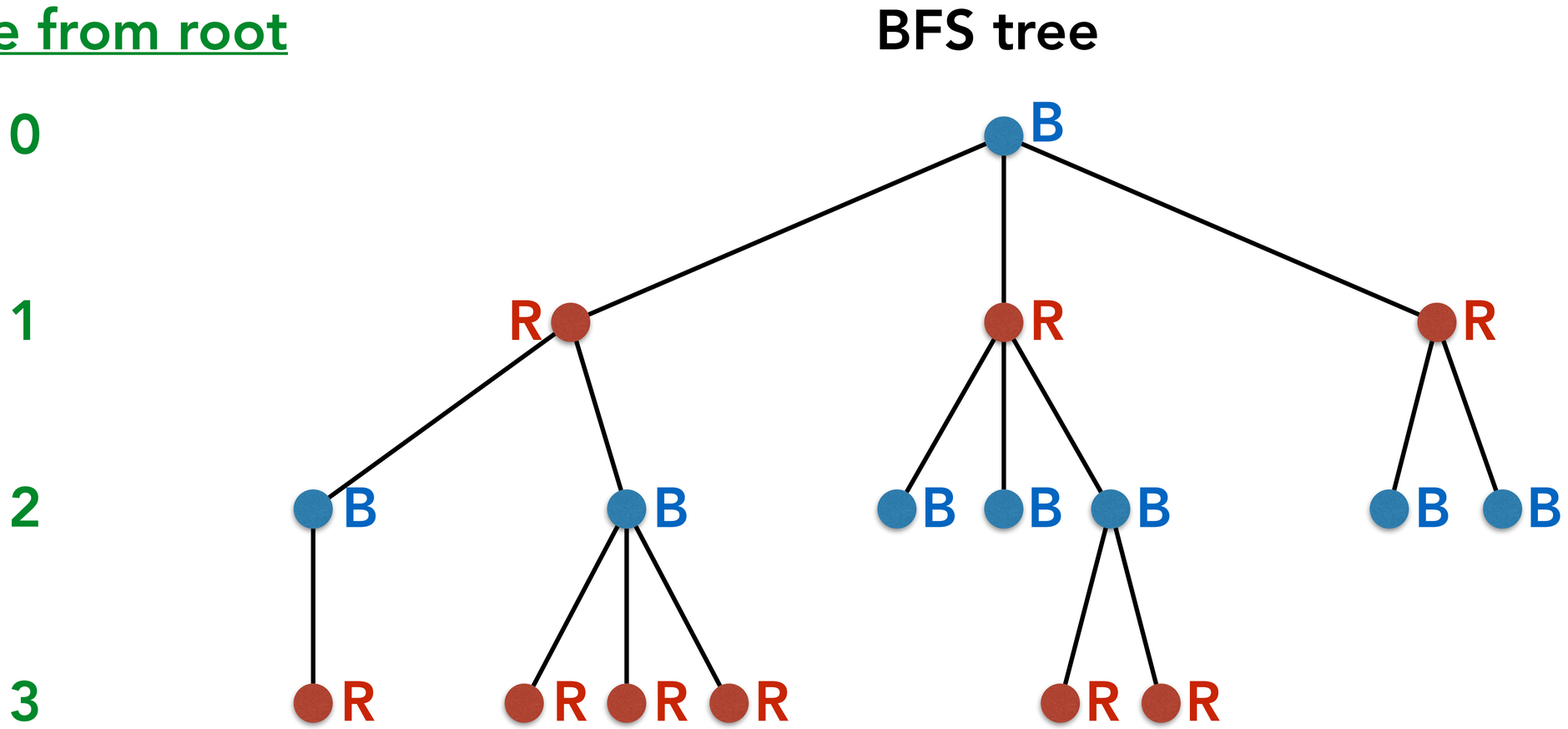
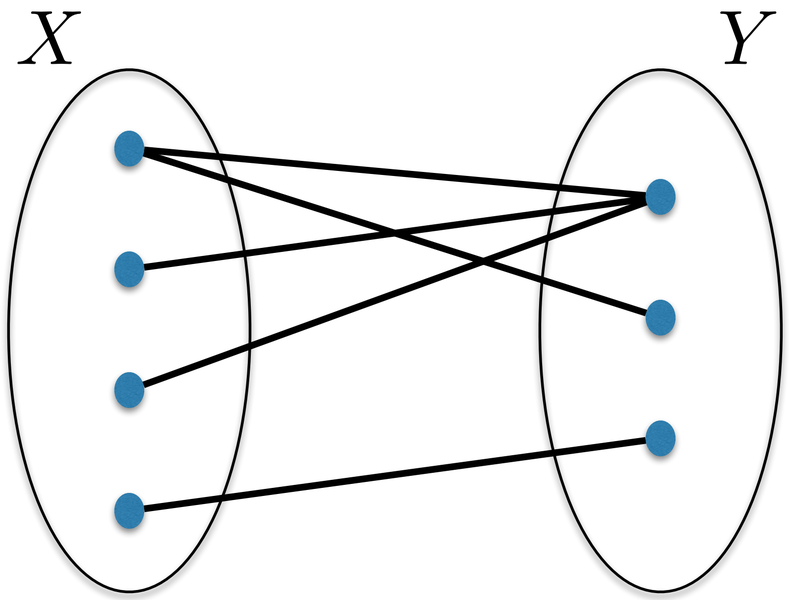


Bipartite Graphs



Often we write the bipartition explicitly:

$$G = (X, Y, E)$$

Bipartite Graphs

Great at modeling relations between two distinct classes of objects.

Examples:

X = **machines**, Y = **jobs**

An edge $\{x, y\}$ means x is capable of doing y.

X = **professors**, Y = **courses**

An edge $\{x, y\}$ means x can teach y.

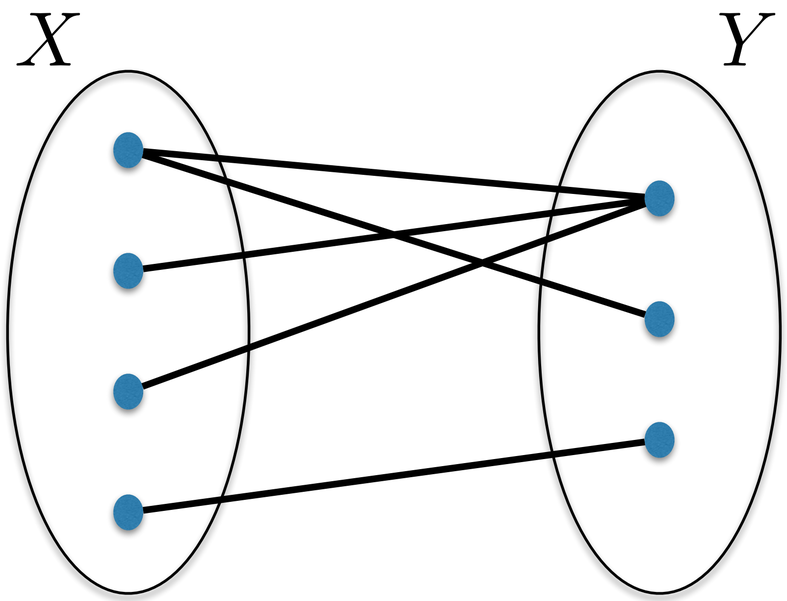
X = **students**, Y = **jobs**

An edge $\{x, y\}$ means x and y are interested in each other.

⋮

Matchings in graphs

Often, we are interested in finding a **matching** in a graph.



Matching:

A subset of the edges that do not share an endpoint.

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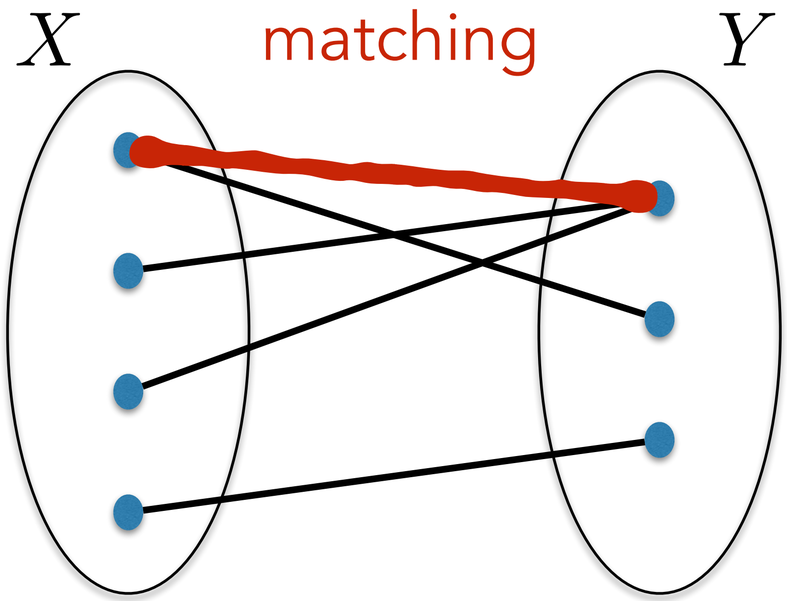

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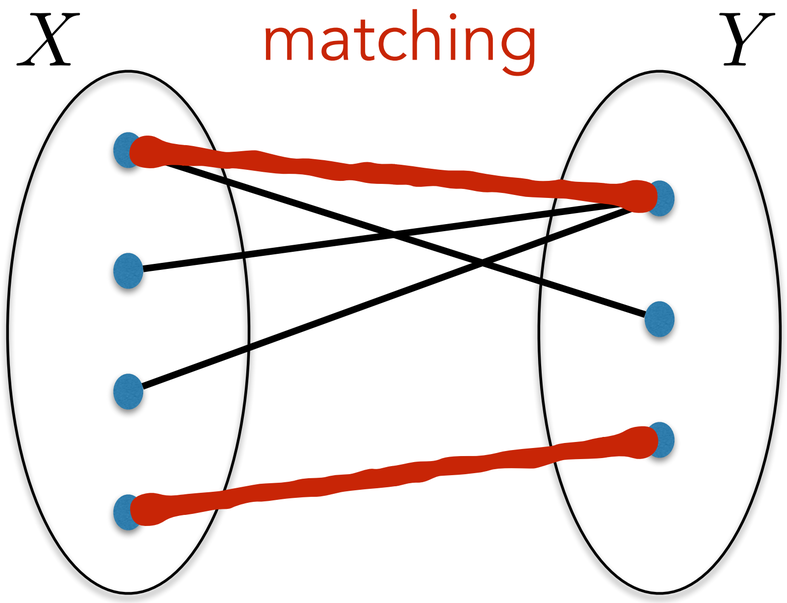

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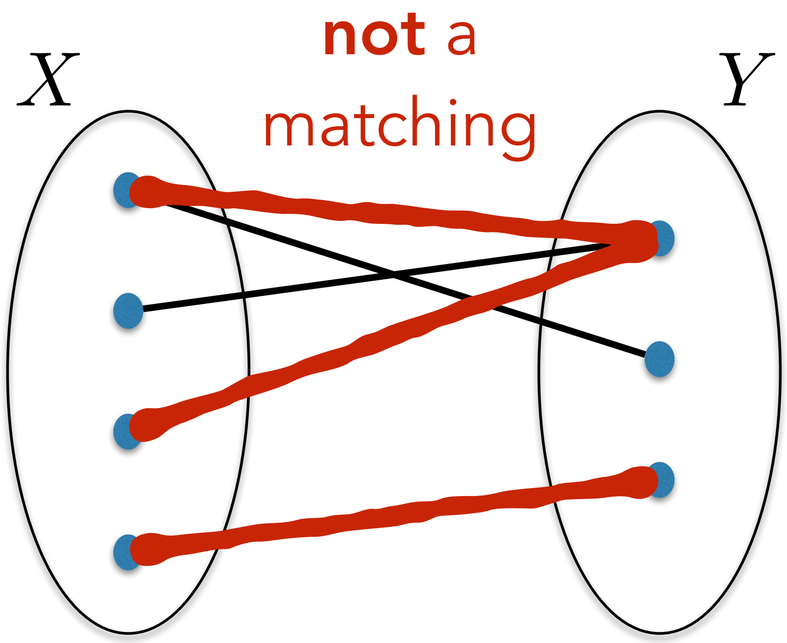

Matching:

A <u>subset of the edges</u> that do not share an endpoint.

Matchings in graphs

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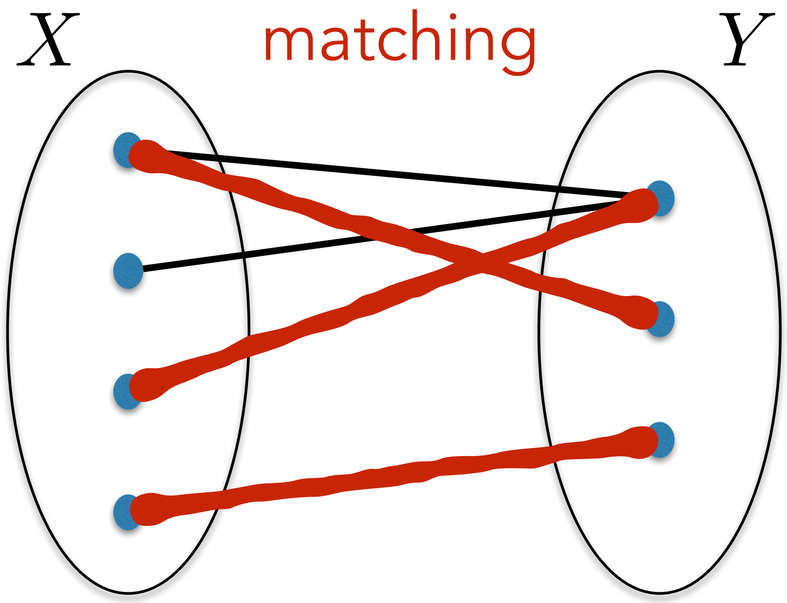

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Matchings in graphs

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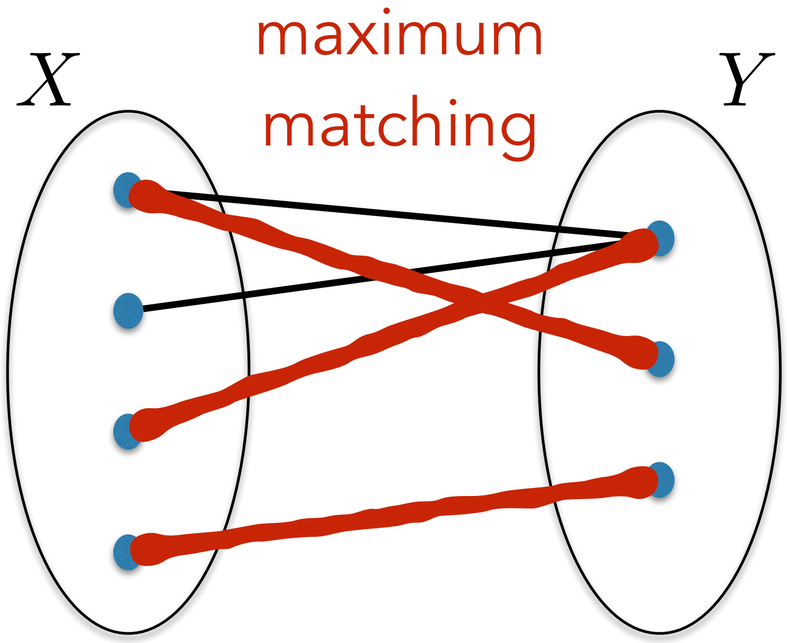


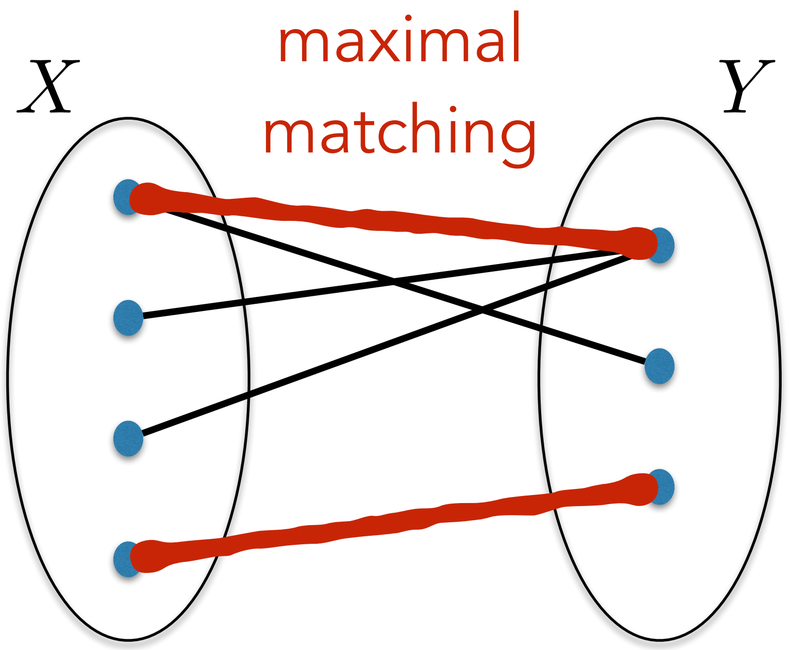
Maximum matching:

A matching with largest number of edges.

(among all possible matchings)

Matchings in graphs

Often, we are interested in finding a **matching** in a graph.



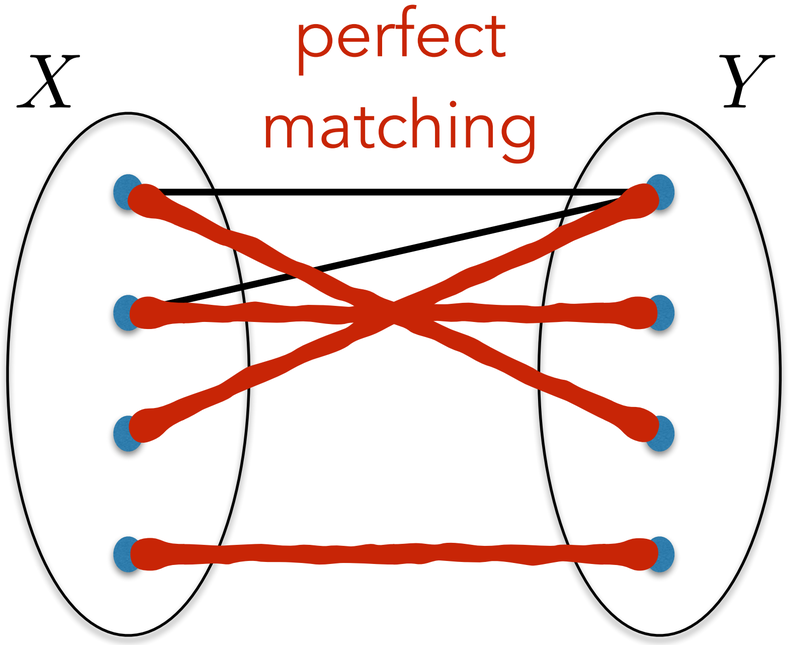
local optimum

Maximal matching:

A matching which cannot contain any more edges.

Matchings in graphs

Often, we are interested in finding a **matching** in a graph.



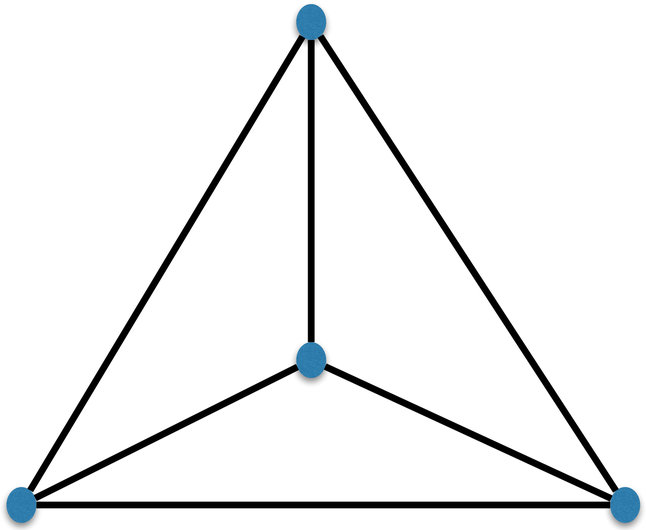
necessary condition for
having a perfect matching
in a bipartite graph:

$$|X| = |Y|$$

Perfect matching:

A matching that covers all vertices.

Matchings are defined for all graphs (not just bipartite).



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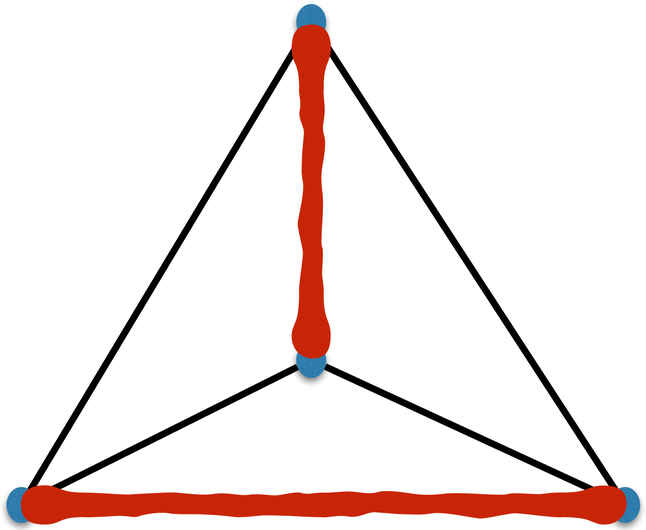


Maximum matching problem

The problem we want to solve is:

Maximum matching problem

Input: A graph $G = (V, E)$.

Output: A maximum matching in G.

Maximum matching problem

Actually, we want to solve the following restriction:

Bipartite maximum matching problem

Input: A bipartite graph $G = (X, Y, E)$.

Output: A maximum matching in G.

How do you solve a problem like this?

1. Formulate the problem 

2. **Ask**: Is there a trivial algorithm? Find and analyze.

3. **Ask**: Is there a better algorithm? Find and analyze.

Step 2: Is there a trivial algorithm?

Bipartite maximum matching problem

Input: A bipartite graph $G = (X, Y, E)$.

Output: A maximum matching in G.

Is there a (trivial) algorithm to solve this problem?

Try all possible subsets of the edges.

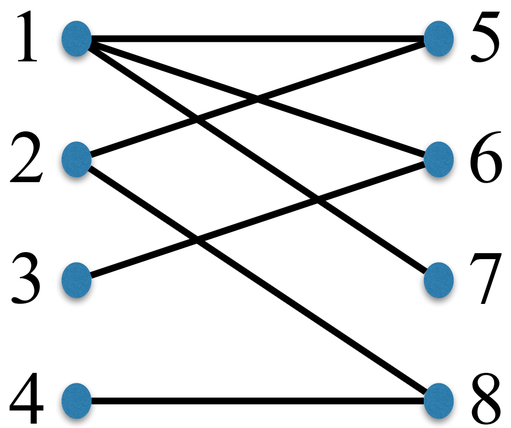
Running time: $\Omega(2^m)$

How do you solve a problem like this?

1. Formulate the problem 
2. **Ask**: Is there a trivial algorithm? Find and analyze. 
3. **Ask**: Is there a better algorithm? Find and analyze.

Trying to find a maximum matching

1st Attempt: What if we picked edges "greedily"?



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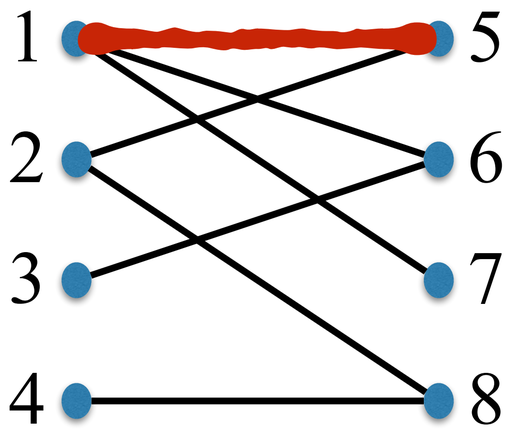

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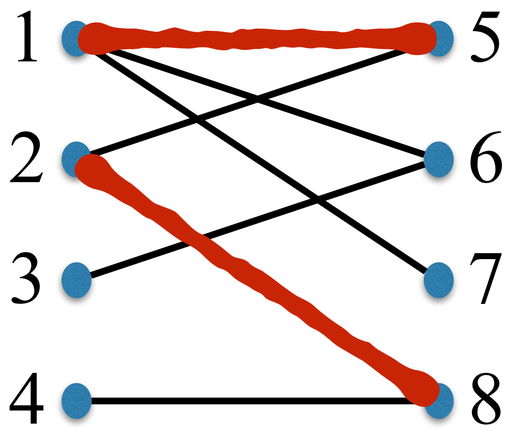

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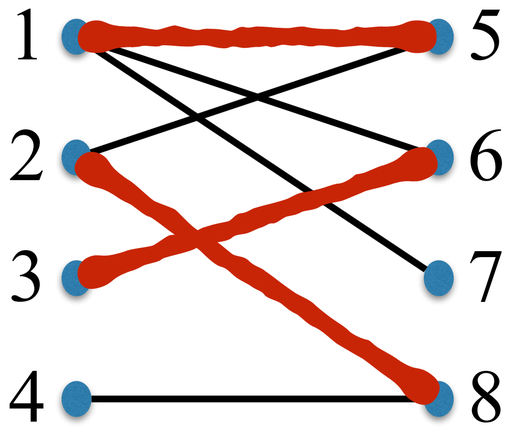

maximal matching

but **not** **maximum**

? Is there a way to get out of this *local optimum*?

What is interesting about the path 4 – 8 – 2 – 5 – 1 – 7 ?

Trying to find a maximum matching

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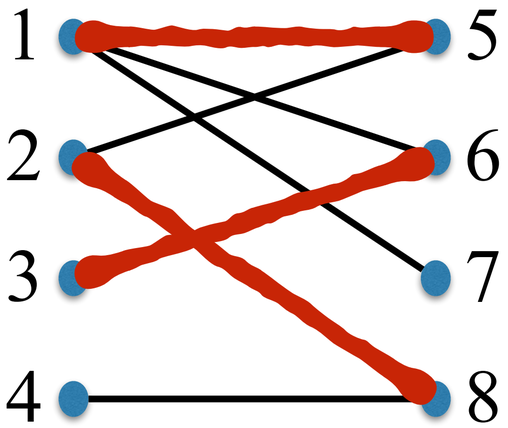
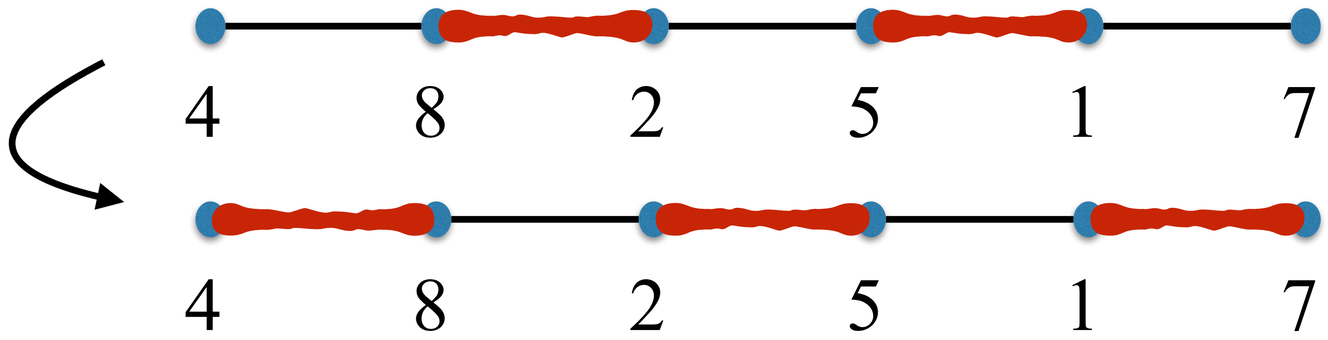


maximal matching

but **not maximum**



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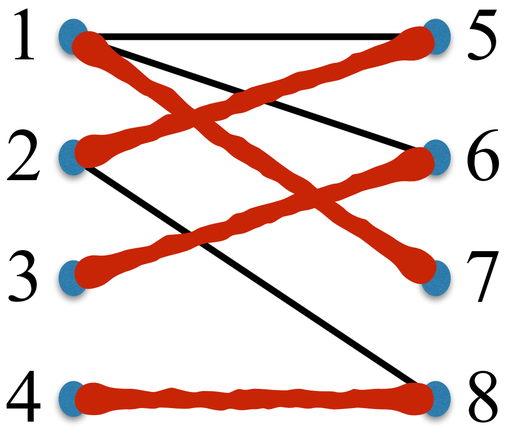
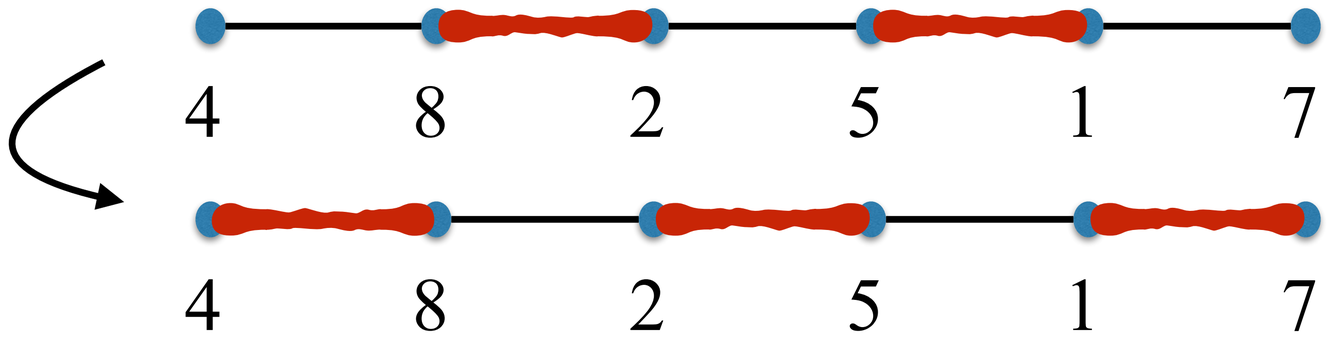


now **maximum**



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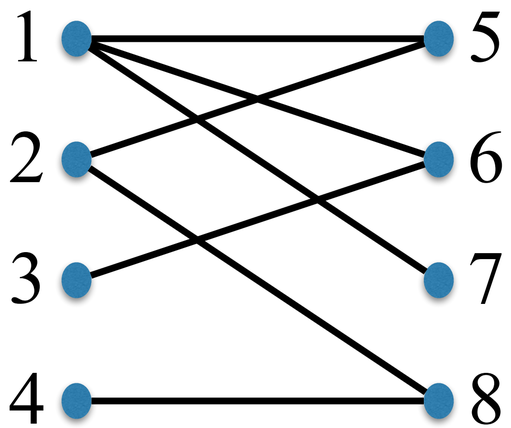

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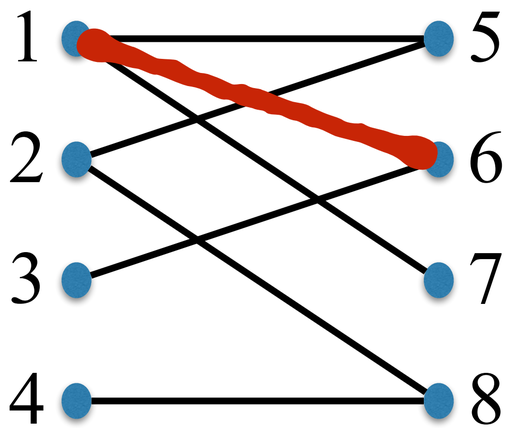

Trying to find a maximum matching

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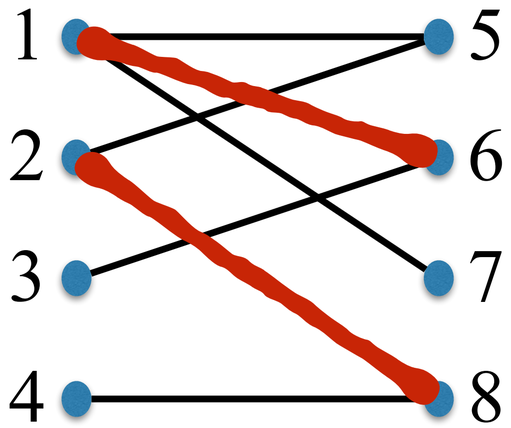


maximal matching

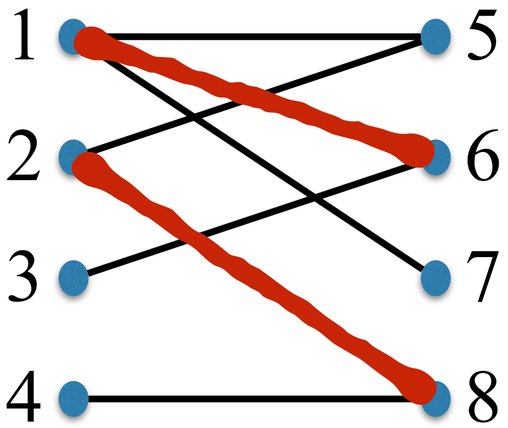
but **not** **maximum**

? Is there a way to get out of this *local optimum*?

What is interesting about the path $5 - 1 - 6 - 3$?

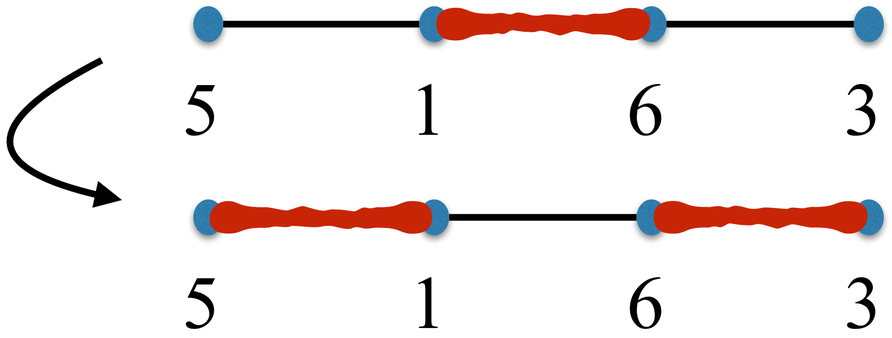
Trying to find a maximum matching

1st Attempt: What if we picked edges "greedily"?



maximal matching

but **not** **maximum**



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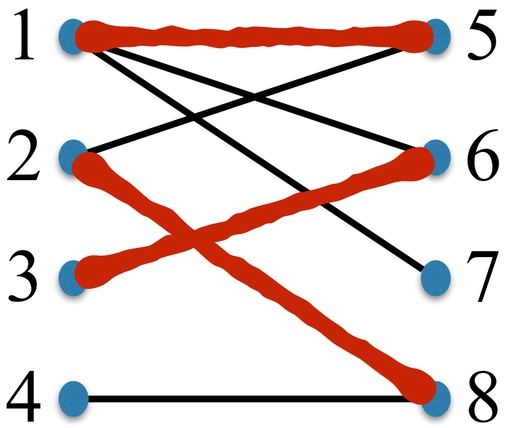
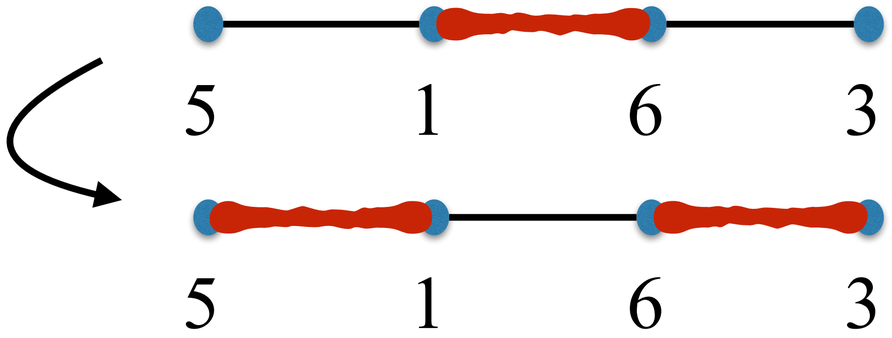


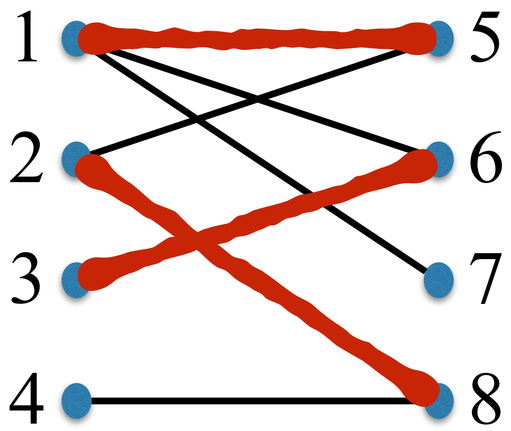
maximal matching

but **not** **maximum**



Trying to find a maximum matching

1st Attempt: What if we picked edges "greedily"?



maximal matching

but **not maximum**

Now same as previous example!

Do the same to find the maximum matching.

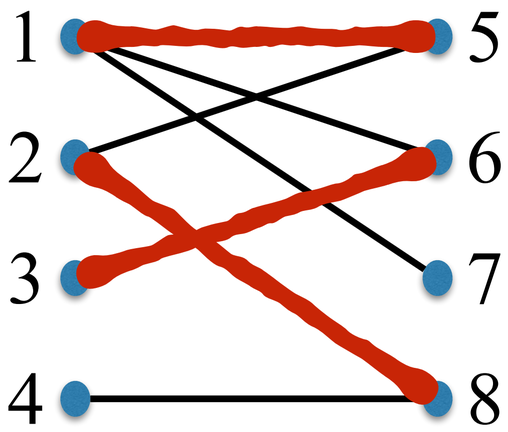
Augmenting path

Let **M** be some matching.

Definition: An **alternating path** with respect to **M** is a path such that edges in the path alternate between being in **M** and not being in **M**.

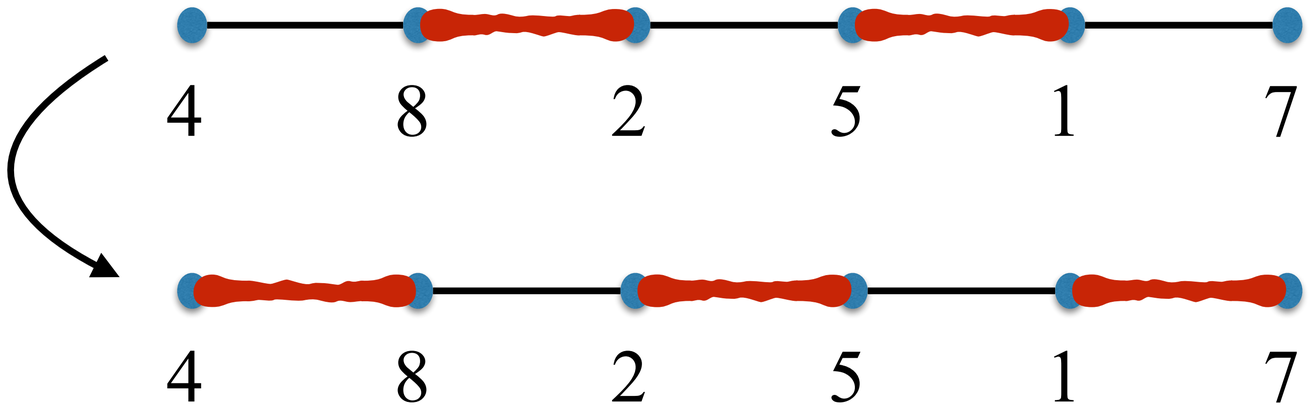
Definition: An **augmenting path** with respect to **M** is an **alternating path** such that the first and last vertices are **not** matched by **M**.

Augmenting path



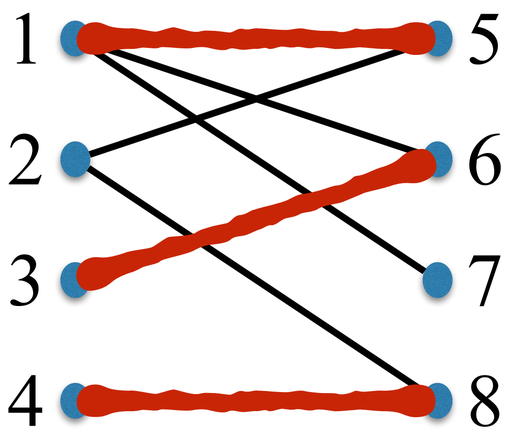
Augmenting path:

4 – 8 – 2 – 5 – 1 – 7



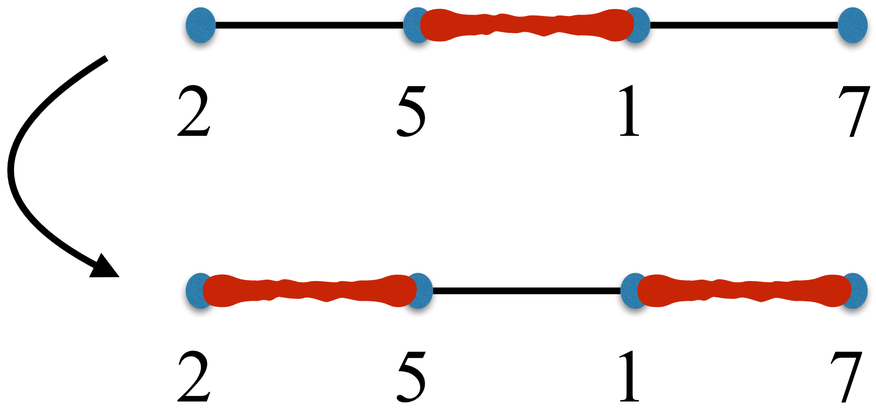
augmenting path $\Longrightarrow$ can obtain a bigger matching.

Augmenting path



Augmenting path:

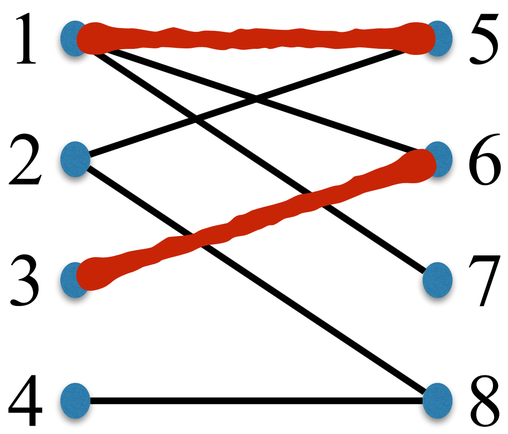
2 – 5 – 1 – 7



An augmenting path
need **not** contain
all the edges of the matching.

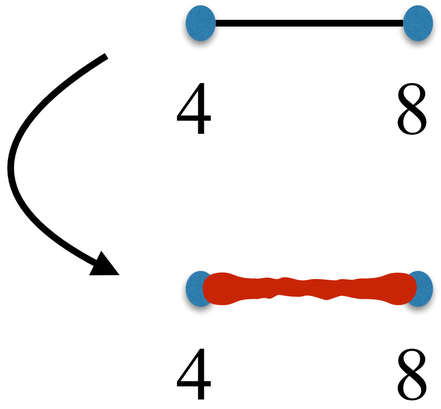
augmenting path $\Longrightarrow$ can obtain a bigger matching.

Augmenting path



Augmenting path:

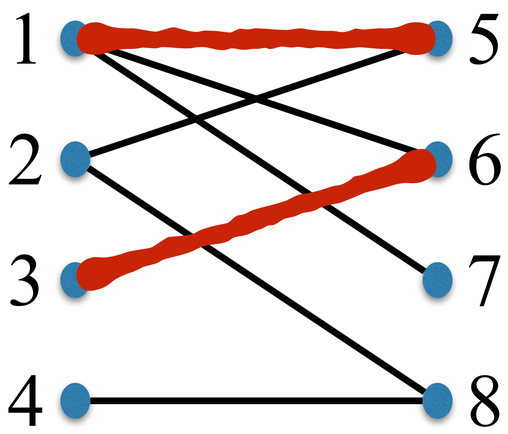
4 – 8



An augmenting path
need **not** contain
any of the edges in the matching.

augmenting path $\Longrightarrow$ can obtain a bigger matching.

Augmenting path



Not (!) an augmenting path:

2 – 5

5 is matched!

Characterization of Maximum Matchings

An obstruction for being a maximum matching:

There is an augmenting path.

Is this the only type of obstruction?

Theorem: A matching **M** is maximum **if and only if** there is **no** augmenting path with respect to **M**.

Algorithm to find maximum matching

Theorem: A matching **M** is maximum **if and only if** there is **no** augmenting path with respect to **M**.

Algorithm to find max matching:

- Start with a single edge as your matching **M**.
- Repeat until there is no augmenting path w.r.t. **M**:
  - Find an augmenting path with respect to **M**.
  - Update **M** according to the augmenting path.

? OK, but how do you find an augmenting path?

Proof of theorem

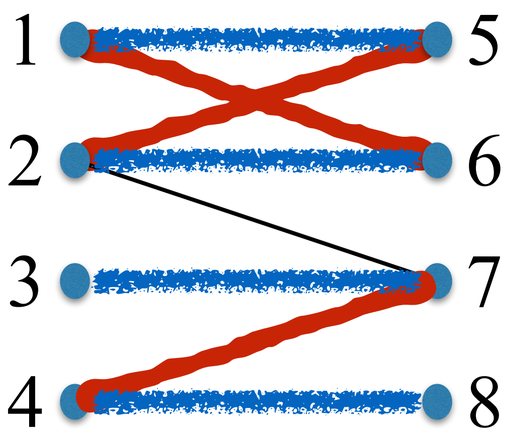
M is not maximum **iff** ∃ augmenting path w.r.t. **M**

If ∃ an augmenting path w.r.t. **M**,
can find a bigger matching (i.e. **M** is not maximum).

Goal:

If **M** is not maximum, find an augmenting path w.r.t. **M**.

Let **M*** be a matching such that |**M***| > |**M**|.



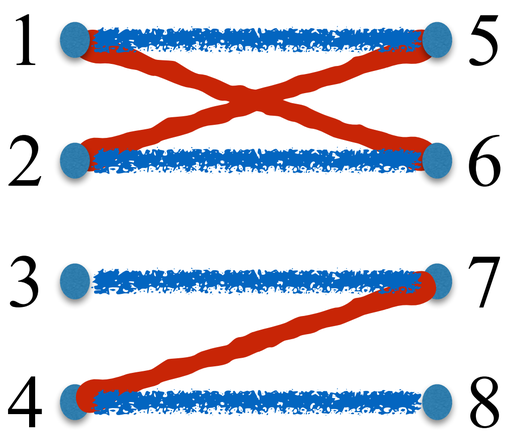
red = **M**

blue = **M***
(horizontal edges)

S = (**M*** ∪ **M**) - (**M** ∩ **M***)

(will find an **augmenting path** in **S**)

Proof of theorem



red = **M**

blue = **M***
(horizontal edges)

S = (**M*** ∪ **M**) - (**M** ∩ **M***)

(will find an **augmenting path** in **S**)

? What does **S** look like?

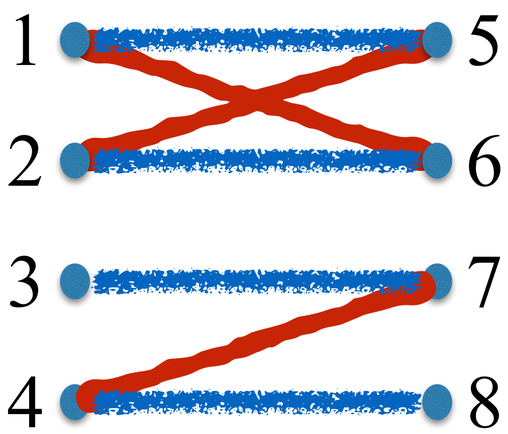
(i) Each vertex has degree 1 or 2. (why?)

(ii) So **S** is a collection of disjoint **cycles** and **paths**. (exercise)

(iii) The edges alternate **red** and **blue**.

(so cycles must have even length)

Proof of theorem



red = **M**

blue = **M***
(horizontal edges)

S = (**M*** ∪ **M**) - (**M** ∩ **M***)

(will find an **augmenting path** in **S**)

So **S** is a collection of disjoint **cycles** (even length) and **paths**.

The edges alternate **red** and **blue**.

blue > # red in **S**

blue = # red in **cycles**

So ∃ a **path** with # blue > # red.

This is an **augmenting path** with respect to **M**. ■

Theorem: A matching **M** is maximum **if and only if** there is **no** augmenting path with respect to **M**.

Summary of proof:

$\Longrightarrow$

If there is an augmenting path, not a max matching.

(can find a bigger matching)

$\Longleftarrow$

If **M** is not maximum, $\exists$ **M*** s.t. $|M^*| > |M|$.

Can find an augmenting path w.r.t. **M**

in the "symmetric difference" of **M*** and **M**.

Theorem: A matching **M** is maximum **if and only if** there is **no** augmenting path with respect to **M**.

Algorithm to find max matching:

- Start with a single edge as your matching **M**.
- Repeat until there is no augmenting path w.r.t. **M**:
    - Find an augmenting path with respect to **M**.
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How do you solve a problem like this?

1. Formulate the problem 
2. **Ask**: Is there a trivial algorithm? Find and analyze. 
3. **Ask**: Is there a better algorithm? Find and analyze. 

Next Time: Stable Matchings