Some real-world examples of matching problems:

How do you solve/approach such a problem:

1.

2.

3.

Bipartite Graphs

Definition. A graph $G = (V, E)$ is bipartite if:

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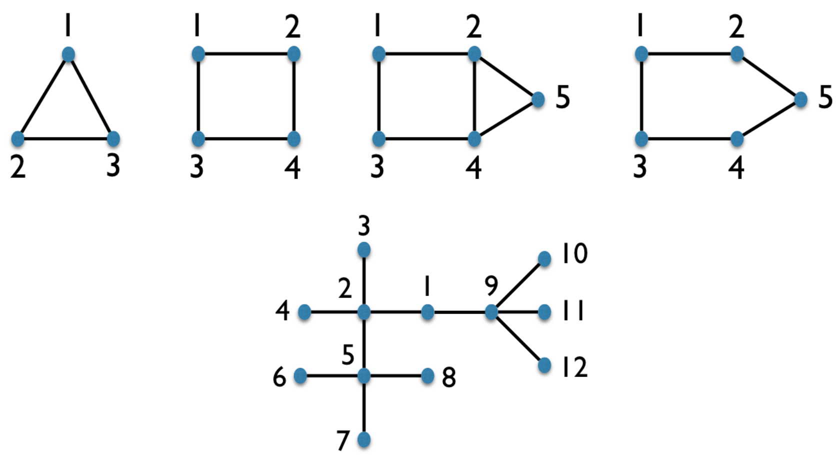
Which of the following graphs are bipartite?



Observation. An obstruction for being bipartite is:

Question. Is this the only type of obstruction?

Theorem.

Proof.

Bipartite graphs are great at modeling relations between two distinct groups of objects.

- X = machines. Y = jobs. An edge $\{x, y\}$ means machine x is capable of doing job y.
- X = professors. Y = courses. An edge $\{x, y\}$ means professor x is capable of teaching course y.
- X = students. Y = internships. An edge $\{x, y\}$ means x and y are interested in each other.
- $\cdots$

Matchings in Graphs

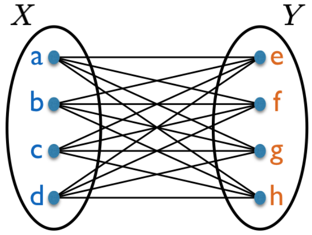
Definition. A matching is:

Definition. A maximum matching is:

Definition. A maximal matching is:

Definition. A perfect matching is:

Question. How many different perfect matchings does the following graph have in terms of $n = |X| = |Y|$?



IMPORTANT.

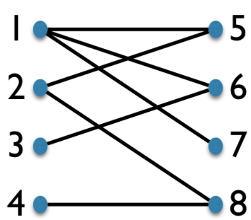
Maximum Matching Problem.
Input:
Output:

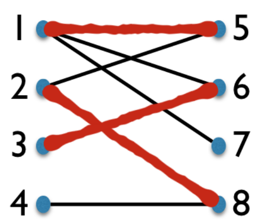
Bipartite Maximum Matching Problem.
Input:
Output:

Question. Is there a trivial algorithm to solve this problem?

Question. Is there a better algorithm to solve this problem?

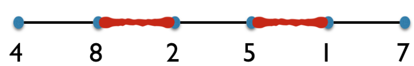
A good first attempt:



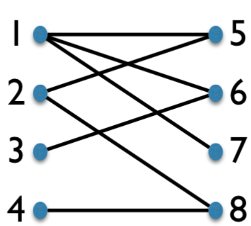


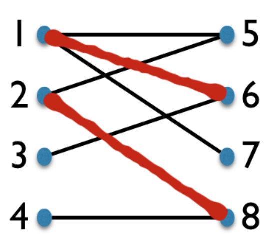
Problem:

Question. Is there a way to get out of this local optimum?
What is interesting about the path $4-8-2-5-1-7$?



Another try:





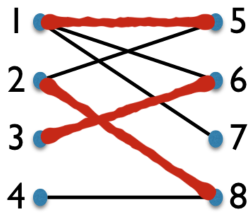
Problem:

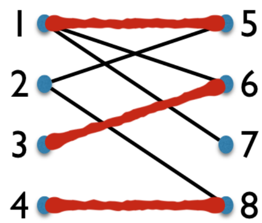
Question. Is there a way to get out of this local optimum?
What is interesting about the path $5-1-6-3$?

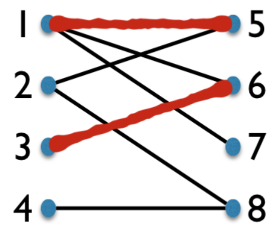
Definition. Let M be some matching in a graph G. An alternating path with respect to M is a path in G such that:

Definition. Let M be some matching in a graph G. An augmenting path with respect to M is:

Examples of augmenting paths.







Observation/Lemma. Let M be some matching in a graph G. If there is an augmenting path with respect to M, then we can find a bigger matching in G (i.e. M is not a maximum matching).

$$\exists \text{ augmenting path w.r.t } M \implies M \text{ not maximum}$$

Theorem.

Algorithm for Bipartite Maximum Matching Problem

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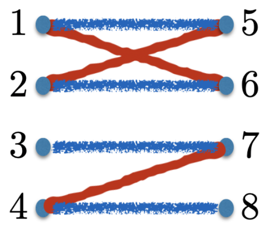
Proof of Theorem

Want to show: If M is not maximum, there is an augmenting path w.r.t. M.



M = a matching that is not maximum.
M^* = a maximum matching.
S = edges contained in M or M^*, but not both.

(will find augmenting path in S)