CS251

Great Ideas
in
Theoretical
Computer Science

Basics of Probability Theory: II

Remember Last Lecture:

The Big Picture

Real World $\longrightarrow$ Mathematical Model

The Big Picture

Real World → Mathematical Model

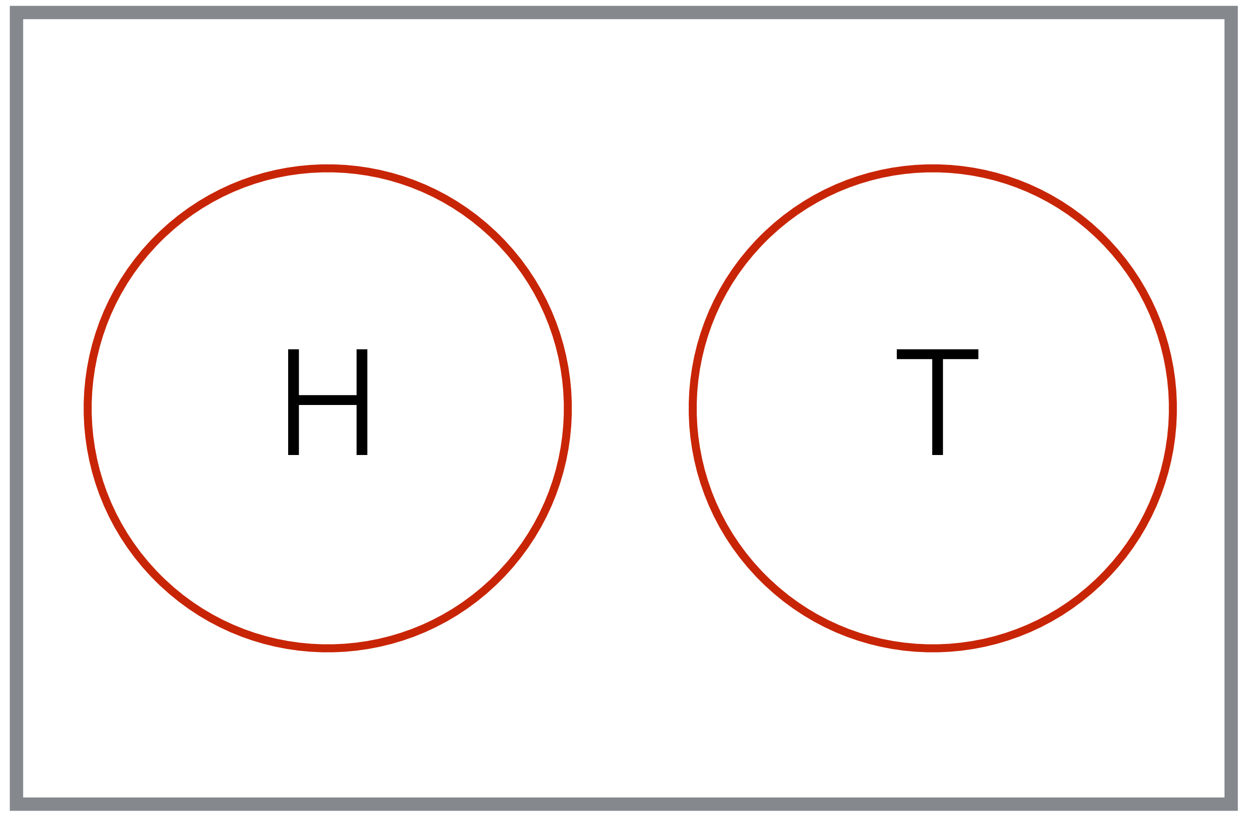
Flip a coin.

The Big Picture

Real World → Mathematical Model

Flip a coin.

Ω



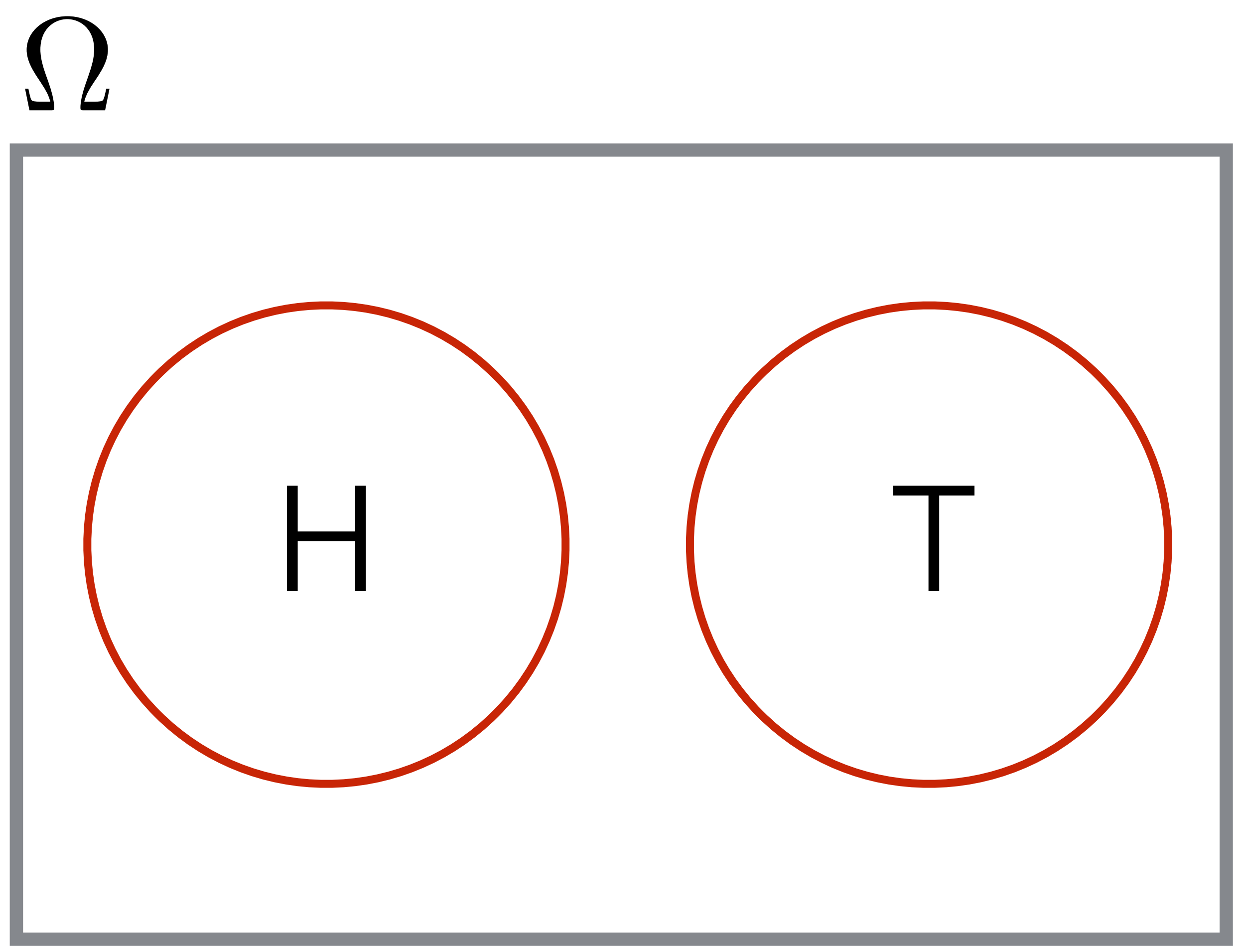
Ω = "sample space"

= set of all possible outcomes

The Big Picture

Real World $\longrightarrow$ Mathematical Model

Flip a coin.



Ω = "sample space"

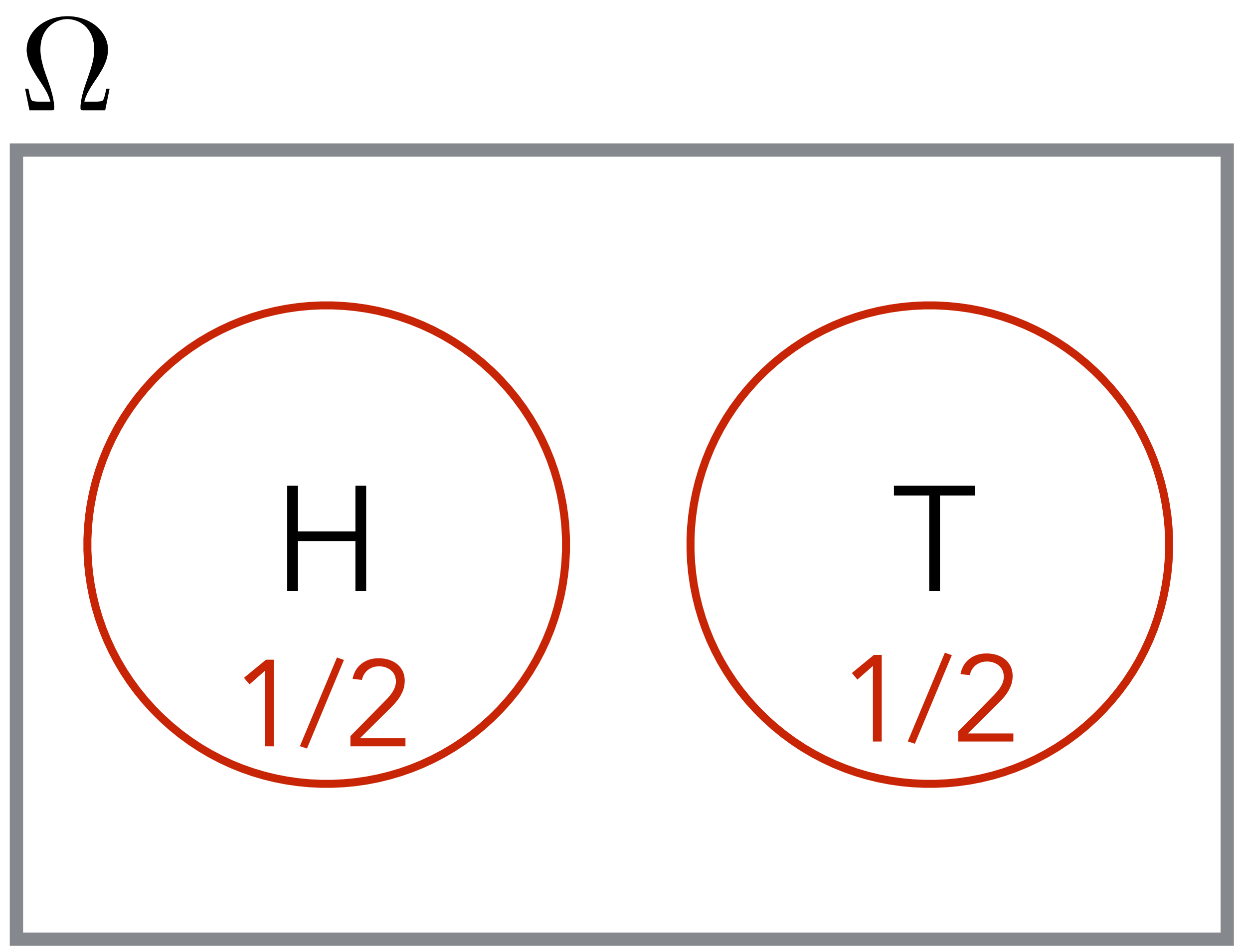
= set of all possible outcomes

$\Pr : \Omega \to [0,1]$ probability distribution

The Big Picture

Real World → Mathematical Model

Flip a coin.



Ω = "sample space"

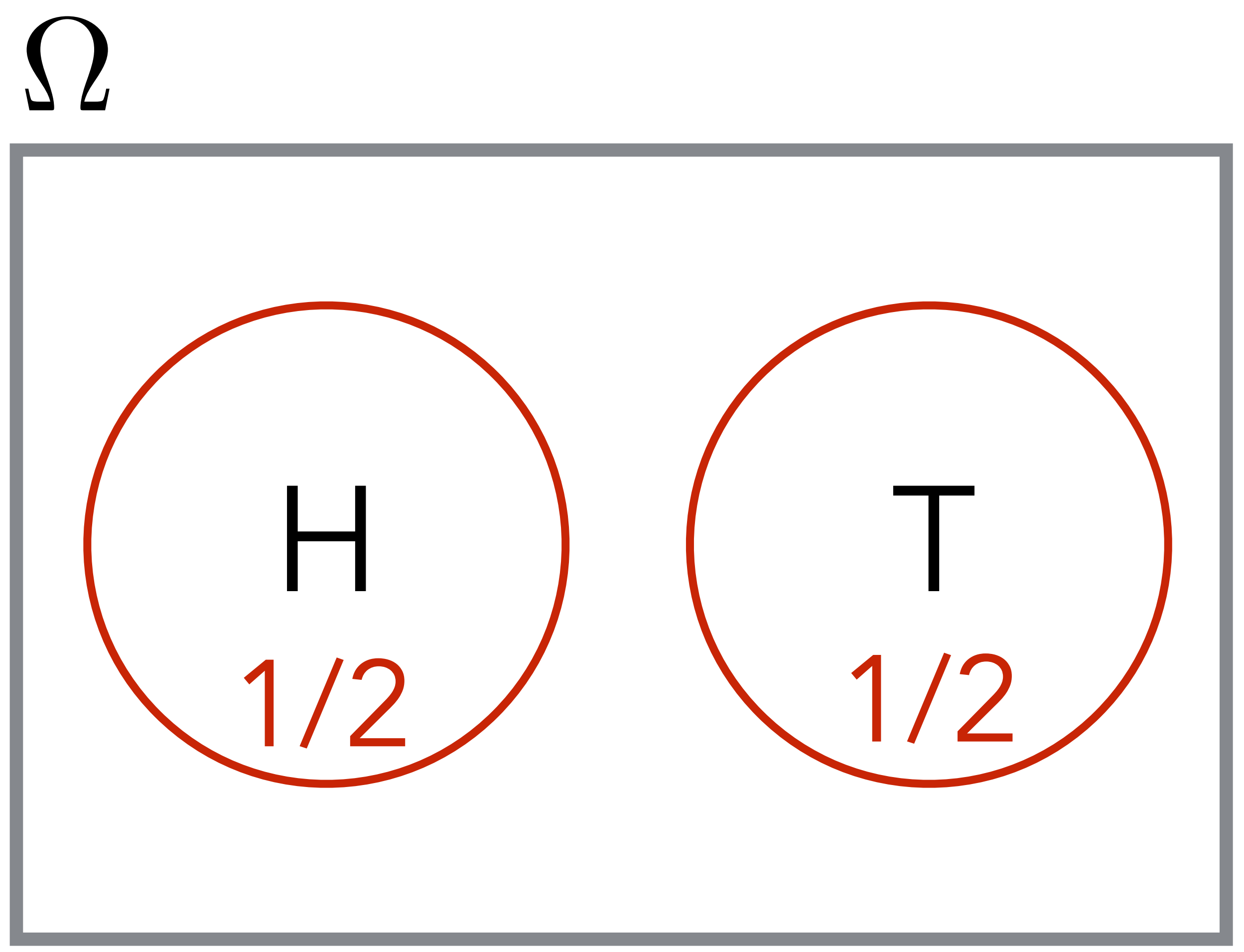
= set of all possible outcomes

$\mathrm{Pr} : \Omega \to [0,1]$ probability distribution

The Big Picture

Real World $\longrightarrow$ Mathematical Model

Flip a coin.



Ω = “sample space”

= set of all possible outcomes

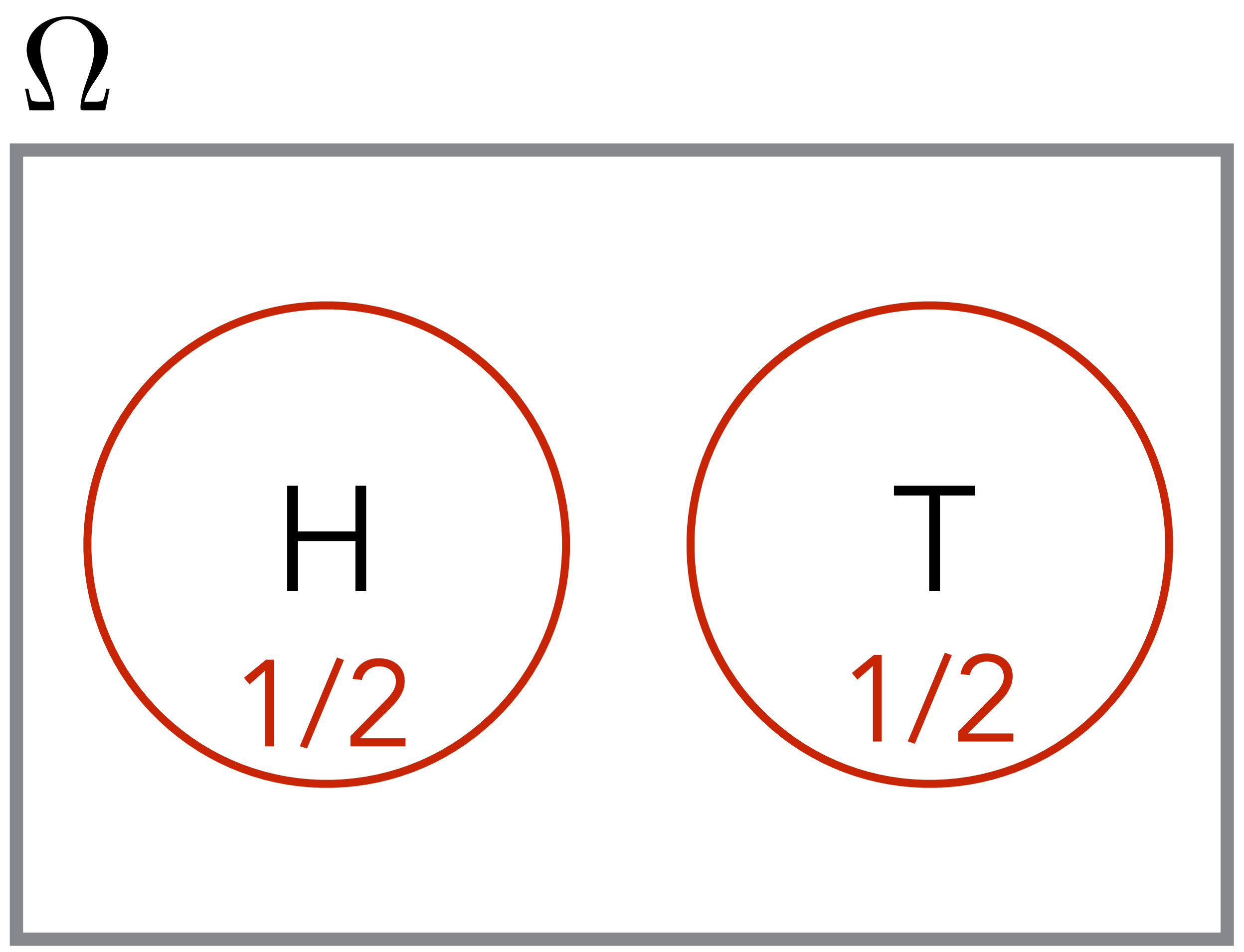
$\Pr : \Omega \to [0, 1]$ probability distribution

$$\sum_{\ell \in \Omega} \Pr[\ell] = 1$$

The Big Picture

Real World $\longrightarrow$ Mathematical Model

Flip a coin.



Ω = "sample space"

= set of all possible outcomes

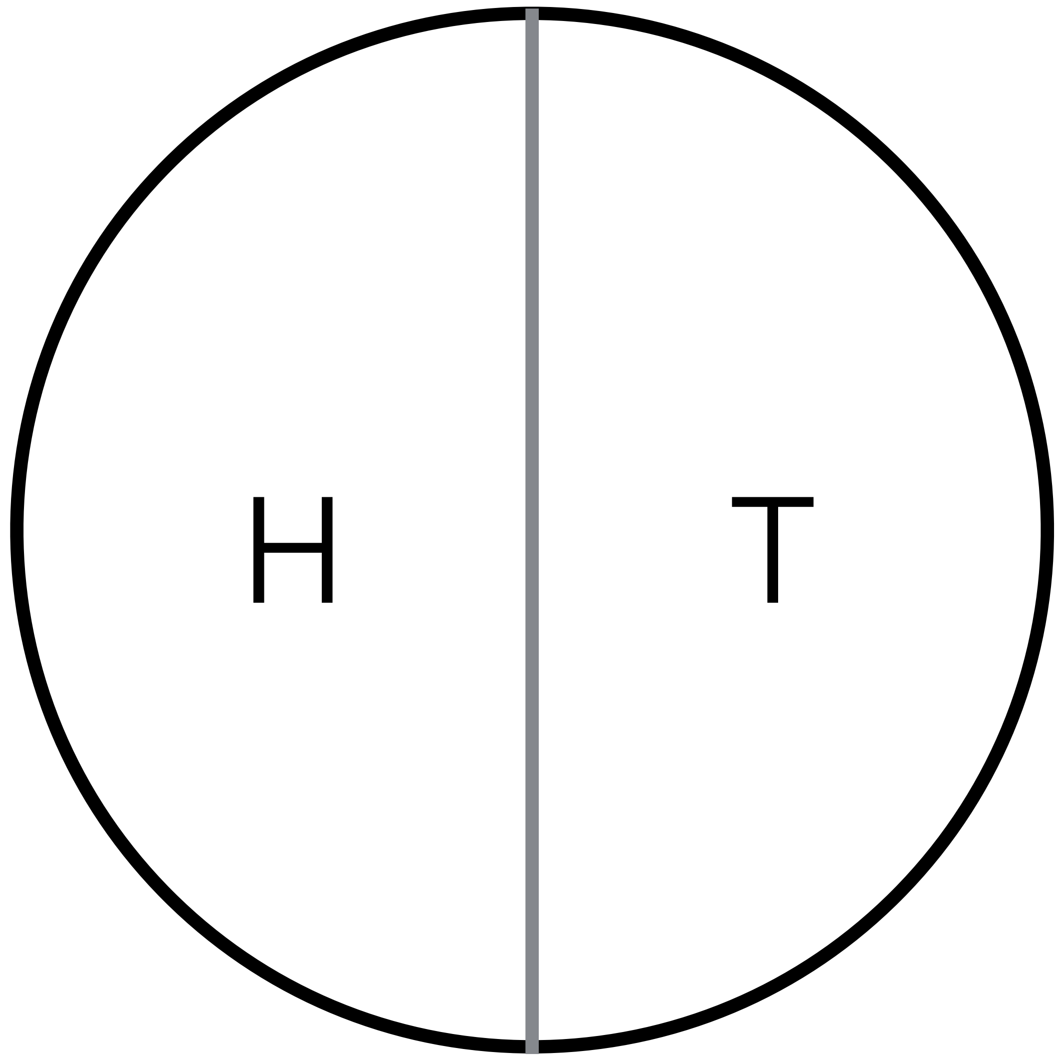
$\Pr : \Omega \to [0,1]$ probability distribution

$$\sum_{\ell \in \Omega} \Pr[\ell] = 1$$ (why?)

The Big Picture

Real World → Mathematical Model

Flip a coin.

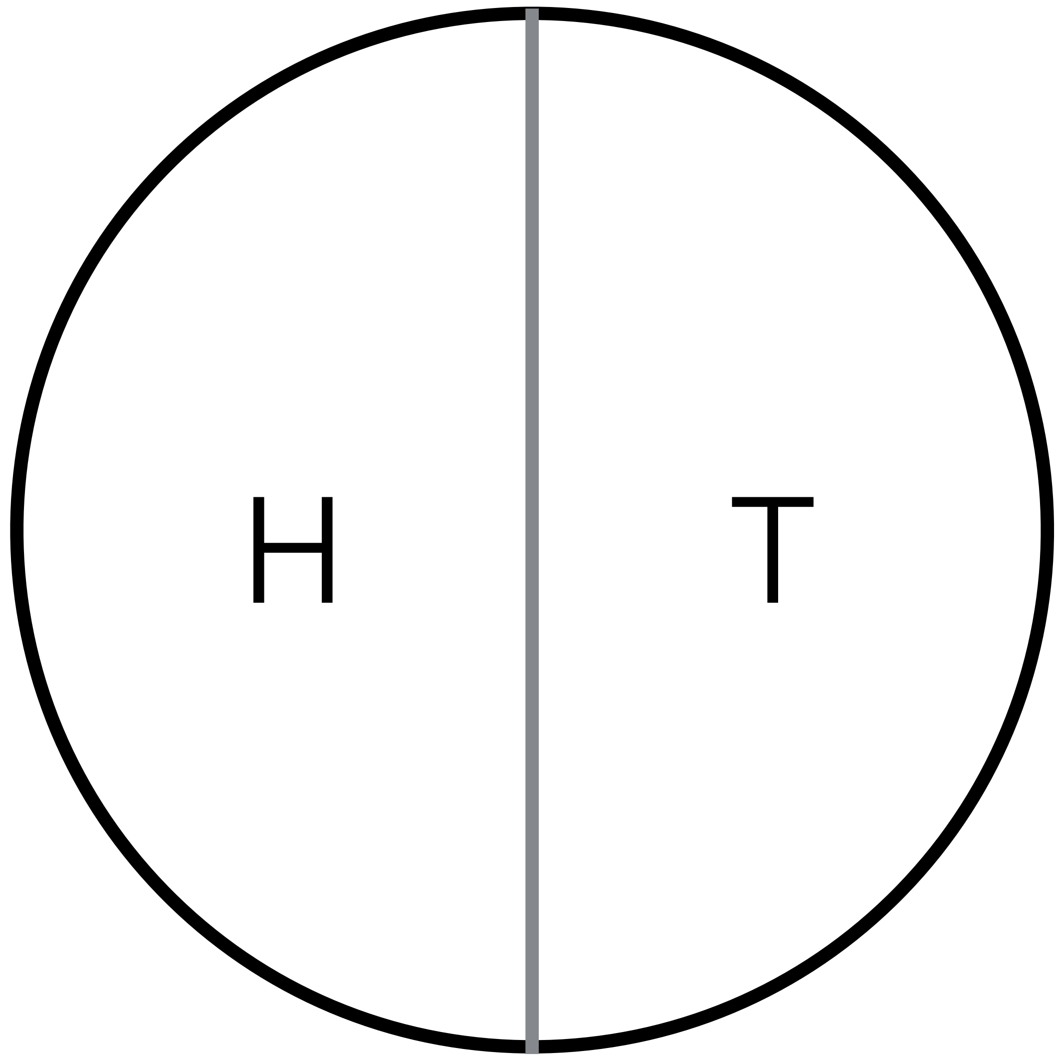


unit pie, **area = 1**

The Big Picture

Real World → Mathematical Model

Flip a coin.



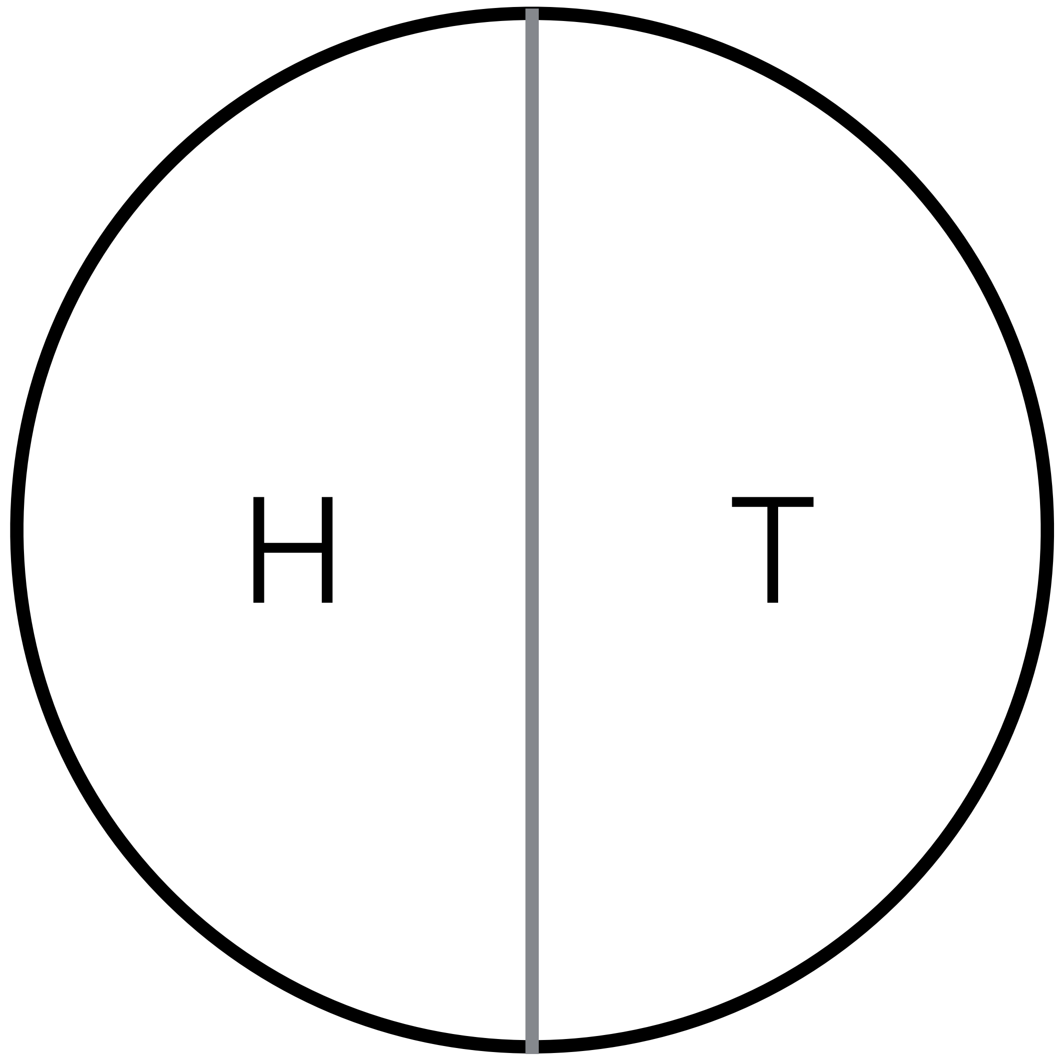
unit pie, **area = 1**

$$\Pr[\text{outcome}] = \text{area of outcome}$$

The Big Picture

Real World → Mathematical Model

Flip a coin.



unit pie, **area = 1**

$$\Pr[\text{outcome}] = \text{area of outcome}$$

$$= \frac{\text{area of outcome}}{\text{area of pie}}$$

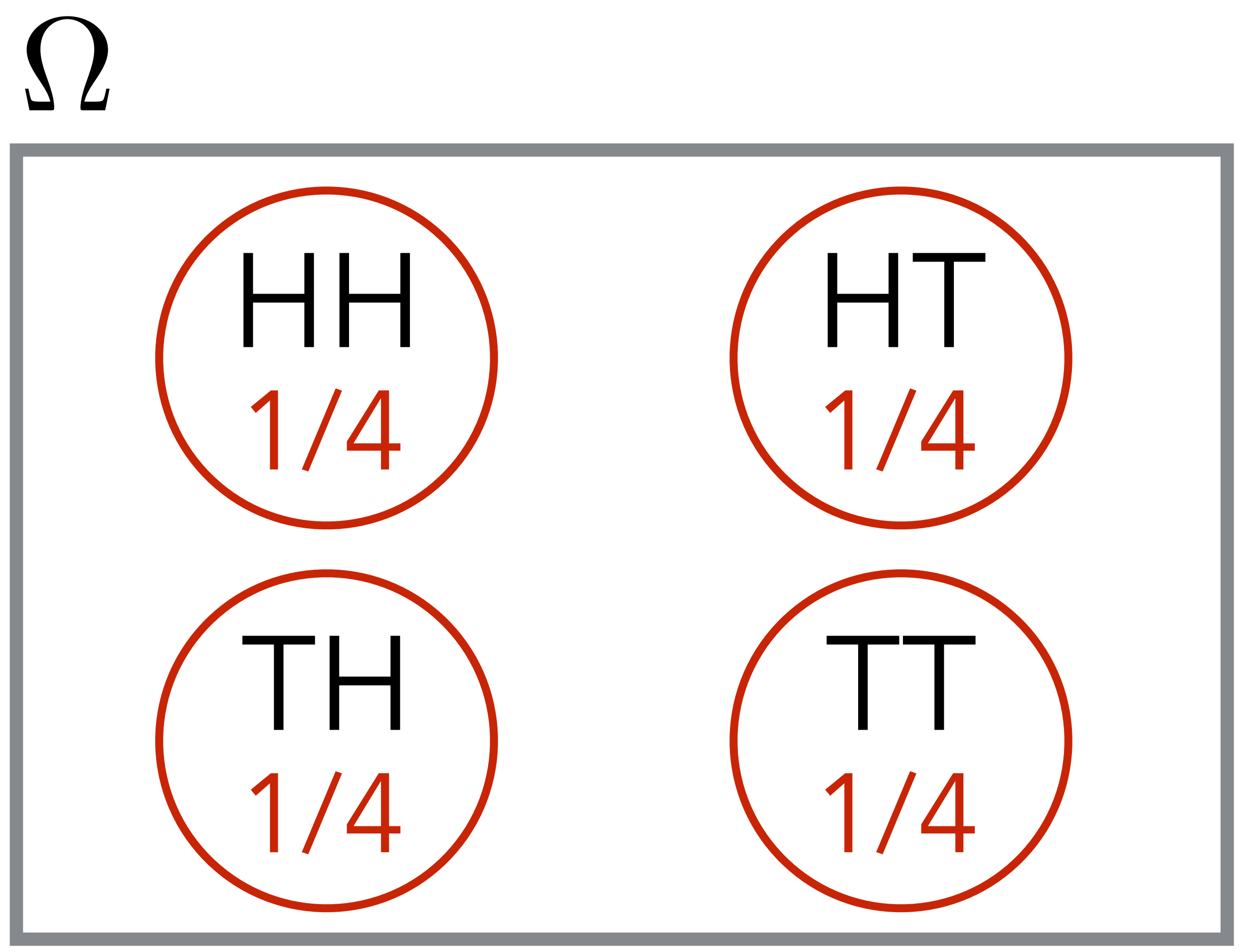
The Big Picture

Real World $\longrightarrow$ Mathematical Model

Flip two coins.

The Big Picture

Real World → Mathematical Model

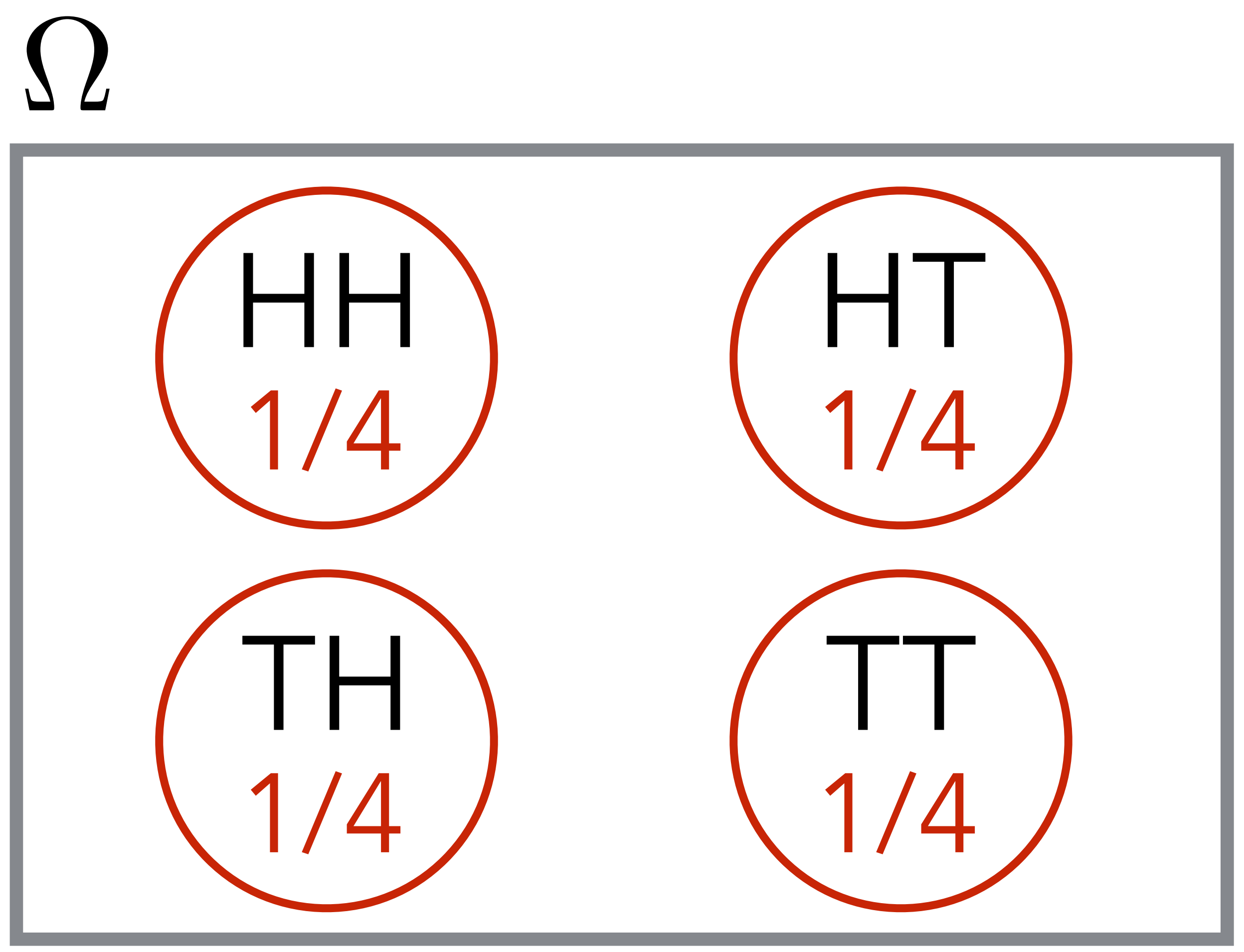


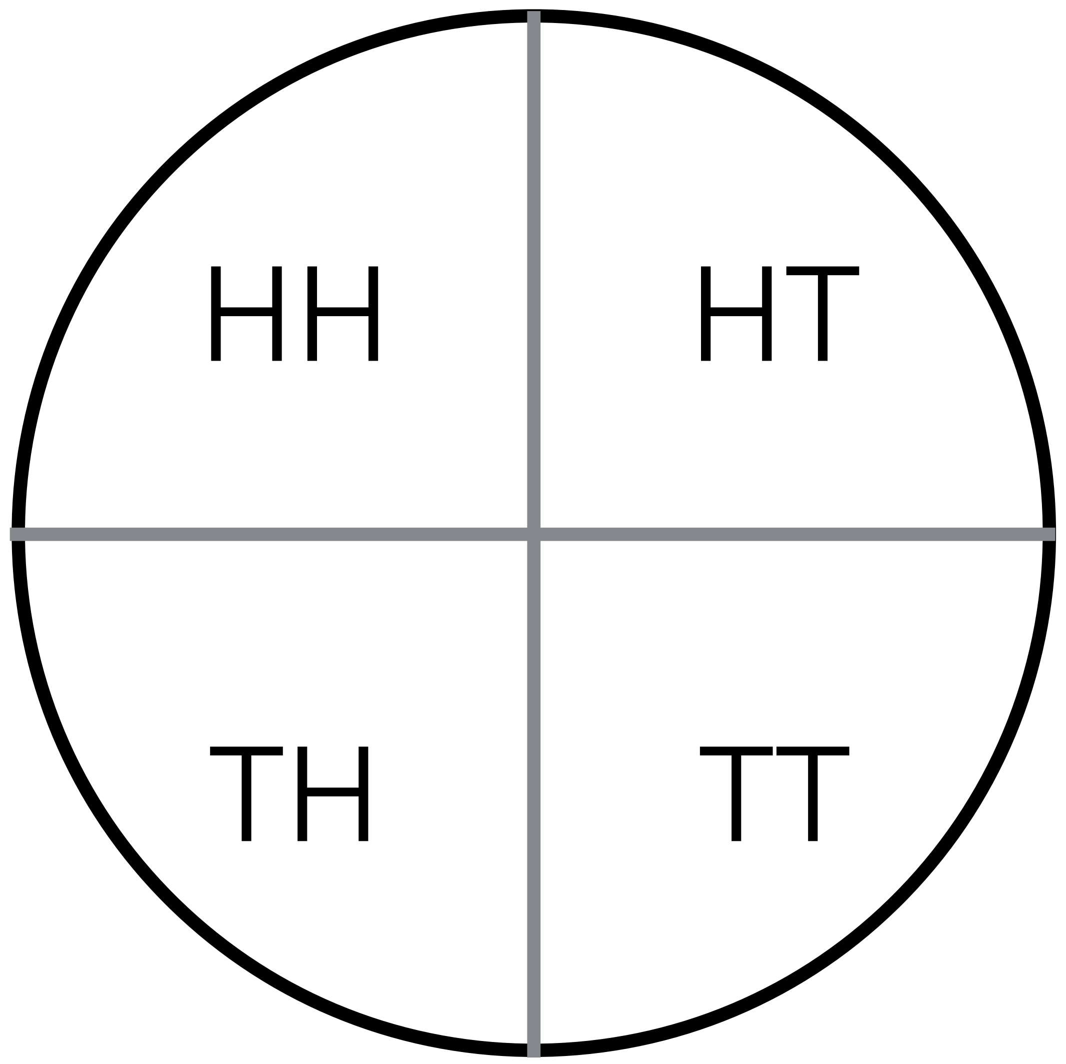
Flip two coins.

The Big Picture

Real World → Mathematical Model

Flip two coins.





The Big Picture

Real World → Code → Probability Tree

The Big Picture

Real World → Code → Probability Tree

Flip a coin.
If it is Heads,
throw a 3-sided die.
If it is Tails,
throw a 4-sided die.

The Big Picture

Real World → Code → Probability Tree

Flip a coin.
If it is Heads,
throw a 3-sided die.
If it is Tails,
throw a 4-sided die.

→

```
flip = Bernoulli(1/2)
if flip == 1: # i.e. Heads
    die = RandInt(3)
else:
    die = RandInt(4)
```

The Big Picture

Real World → Code → Probability Tree

Flip a coin.
If it is Heads,
throw a 3-sided die.
If it is Tails,
throw a 4-sided die.

```
flip = Bernoulli(1/2)
if flip == 1: # i.e. Heads
    die = RandInt(3)
else:
    die = RandInt(4)
```

Probability Tree

Probability Tree

```
flip = Bernoulli(1/2)
if flip == H:
    die = RandInt(3)
else:
    die = RandInt(4)
```

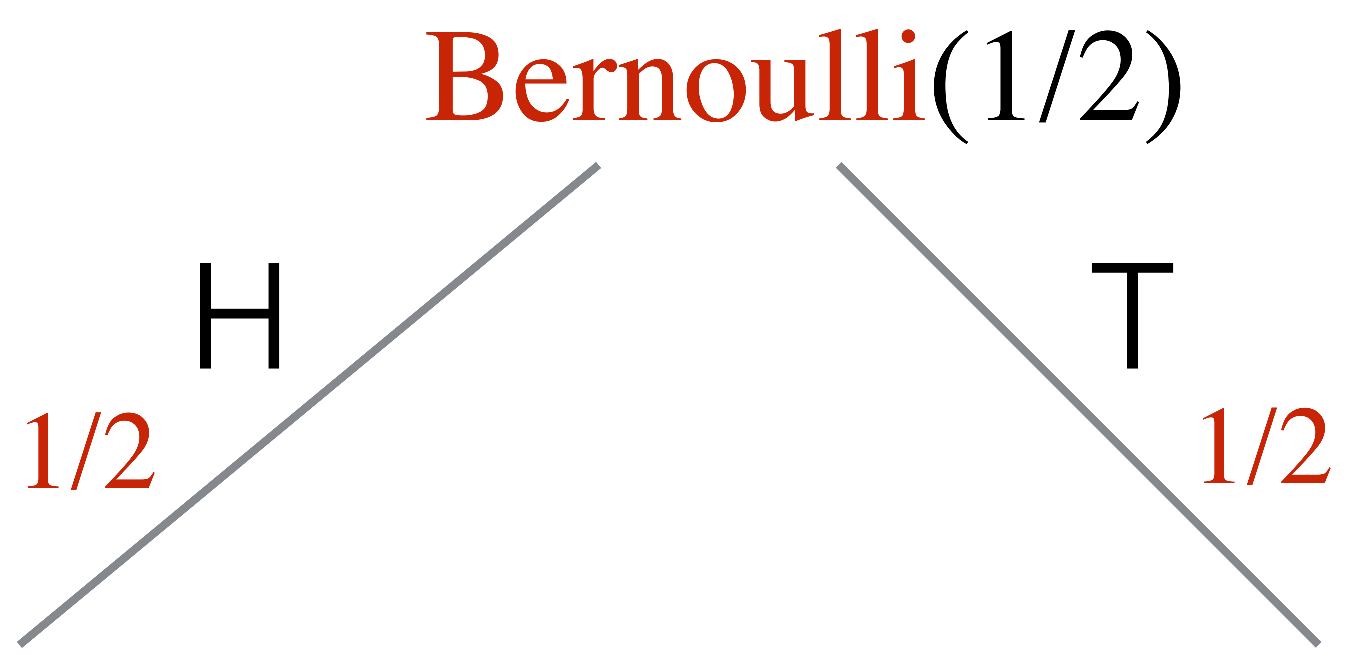
Probability Tree

```
flip = Bernoulli(1/2)
if flip == H:
    die = RandInt(3)
else:
    die = RandInt(4)
```

Bernoulli(1/2)

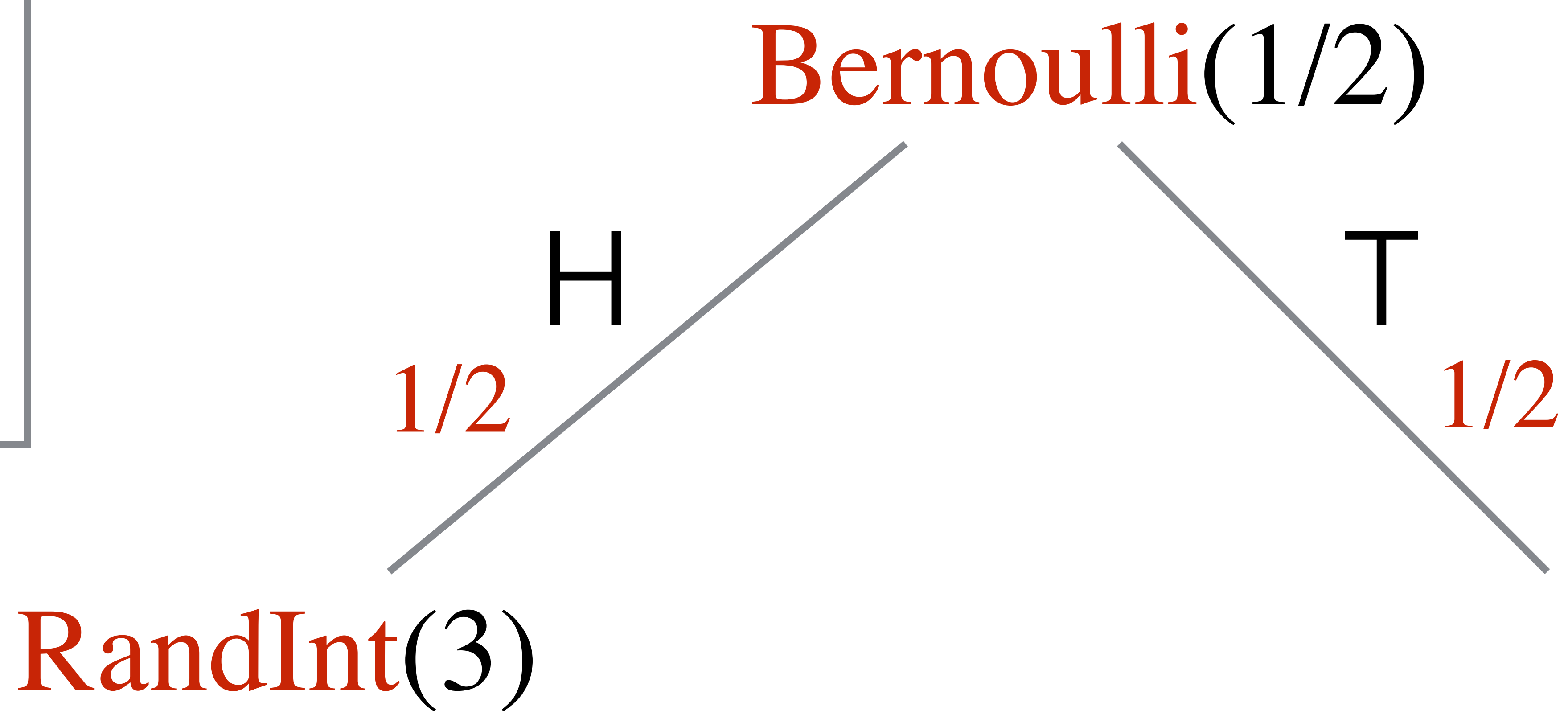
Probability Tree

```
flip = Bernoulli(1/2)
if flip == H:
    die = RandInt(3)
else:
    die = RandInt(4)
```



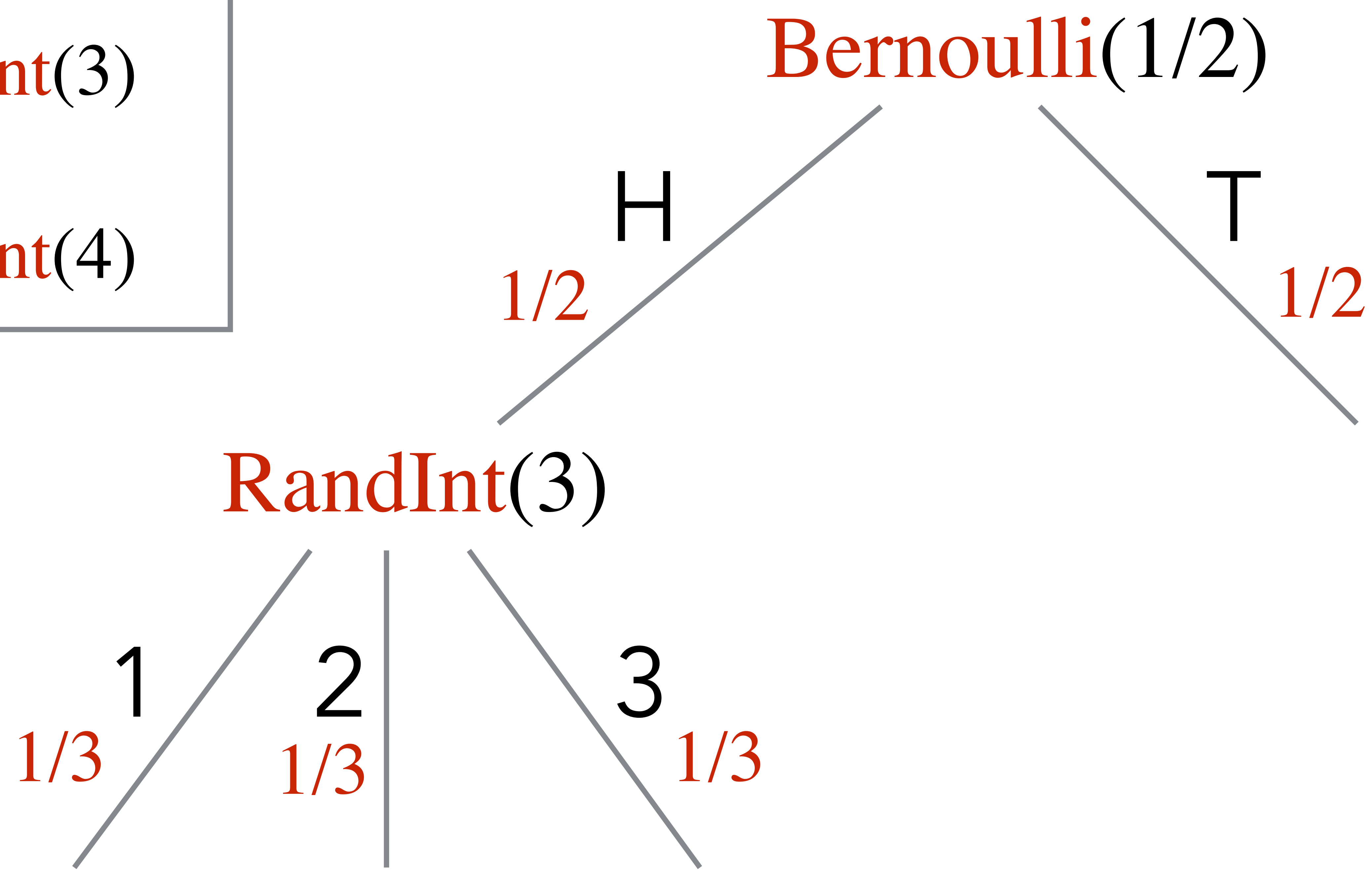
Probability Tree

```
flip = Bernoulli(1/2)
if flip == H:
    die = RandInt(3)
else:
    die = RandInt(4)
```



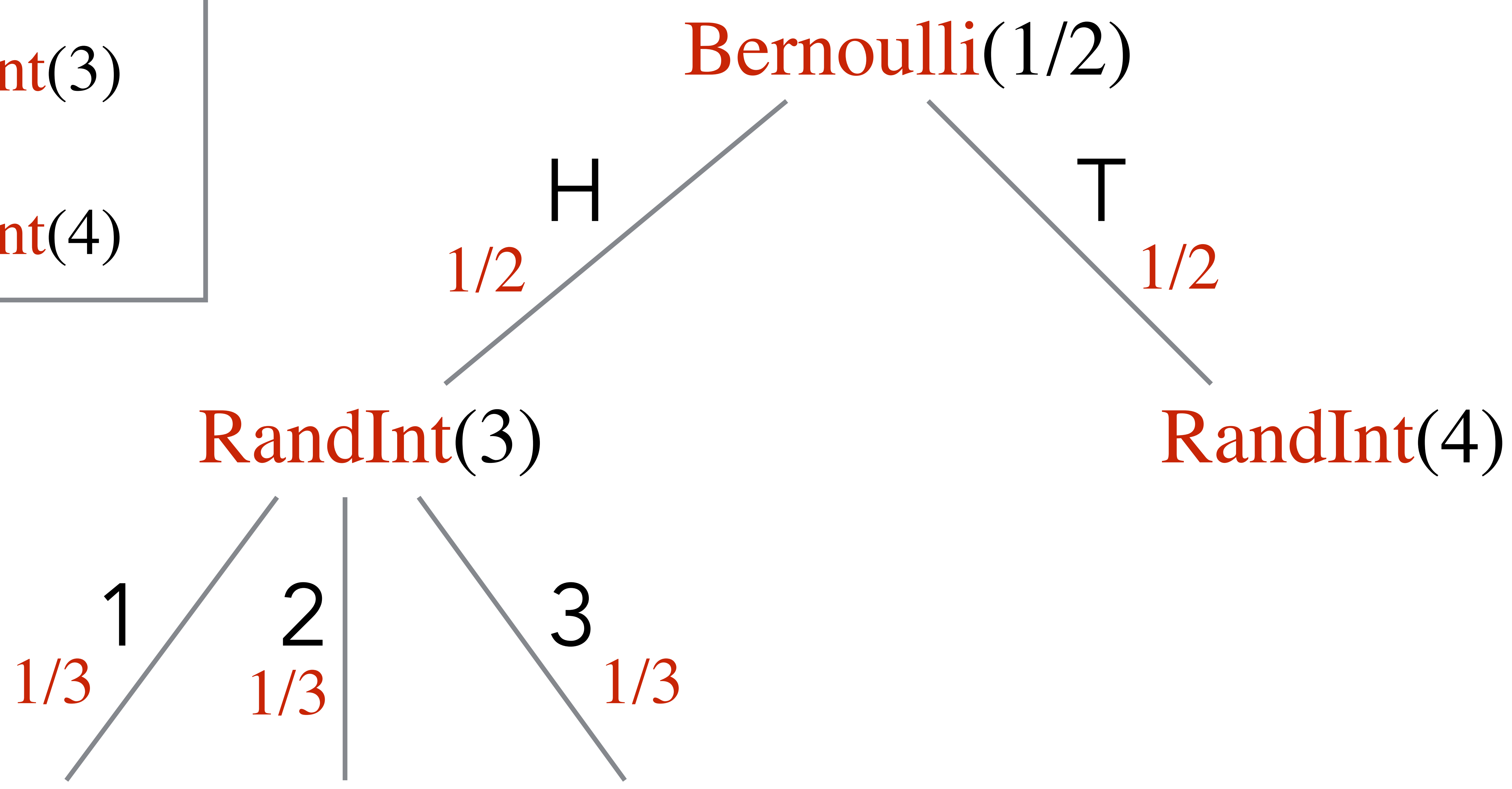
Probability Tree

```
flip = Bernoulli(1/2)
if flip == H:
    die = RandInt(3)
else:
    die = RandInt(4)
```



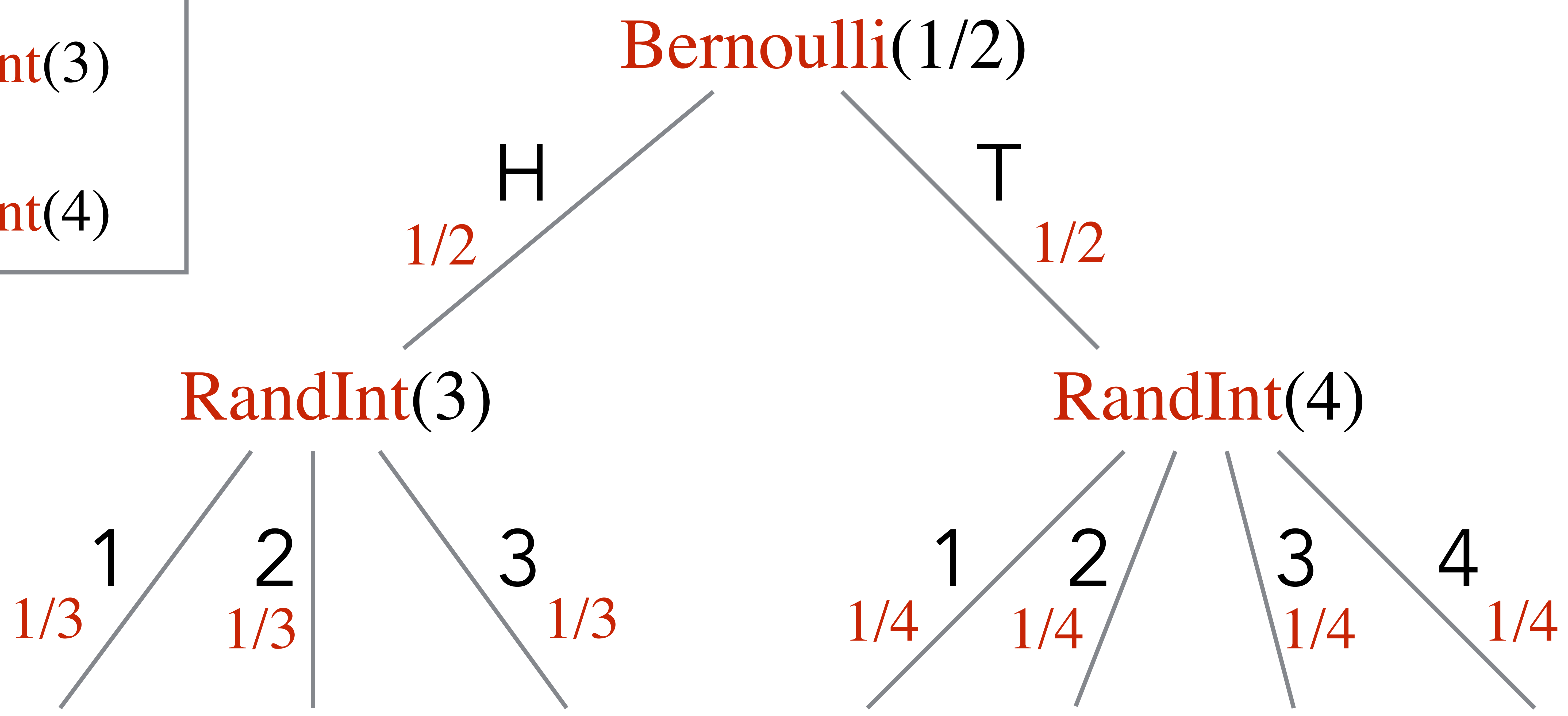
Probability Tree

```
flip = Bernoulli(1/2)
if flip == H:
    die = RandInt(3)
else:
    die = RandInt(4)
```



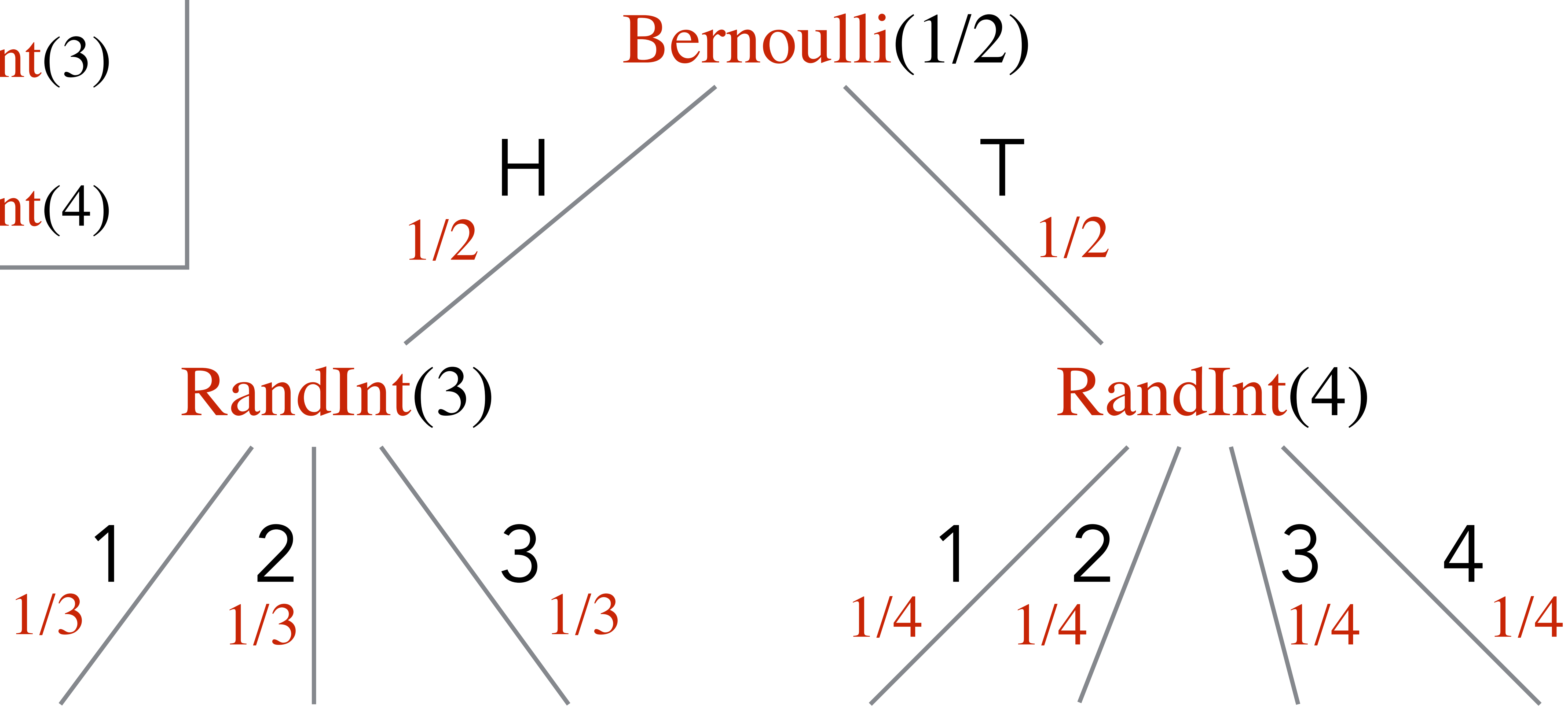
Probability Tree

```
flip = Bernoulli(1/2)
if flip == H:
    die = RandInt(3)
else:
    die = RandInt(4)
```



Probability Tree

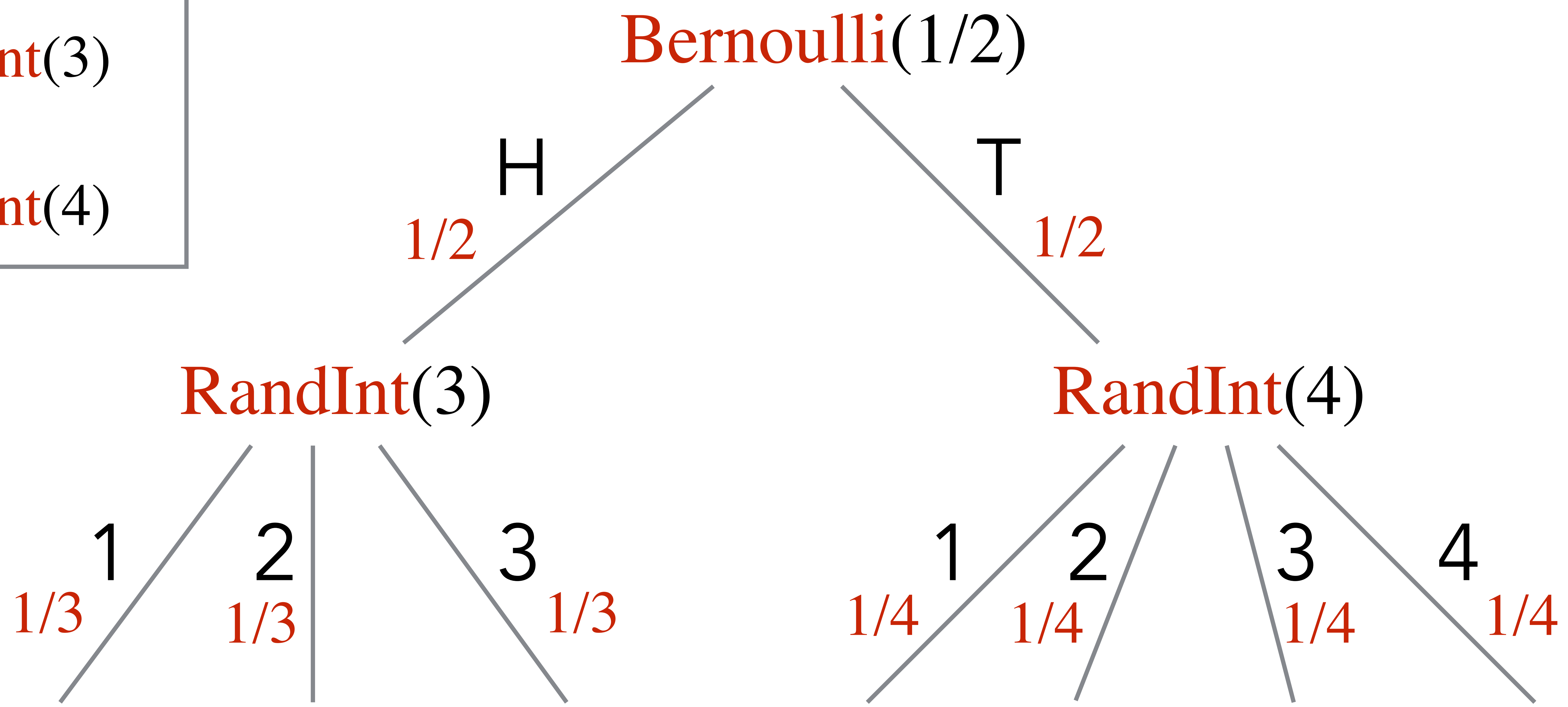
```
flip = Bernoulli(1/2)
if flip == H:
    die = RandInt(3)
else:
    die = RandInt(4)
```



Outcomes:

Probability Tree

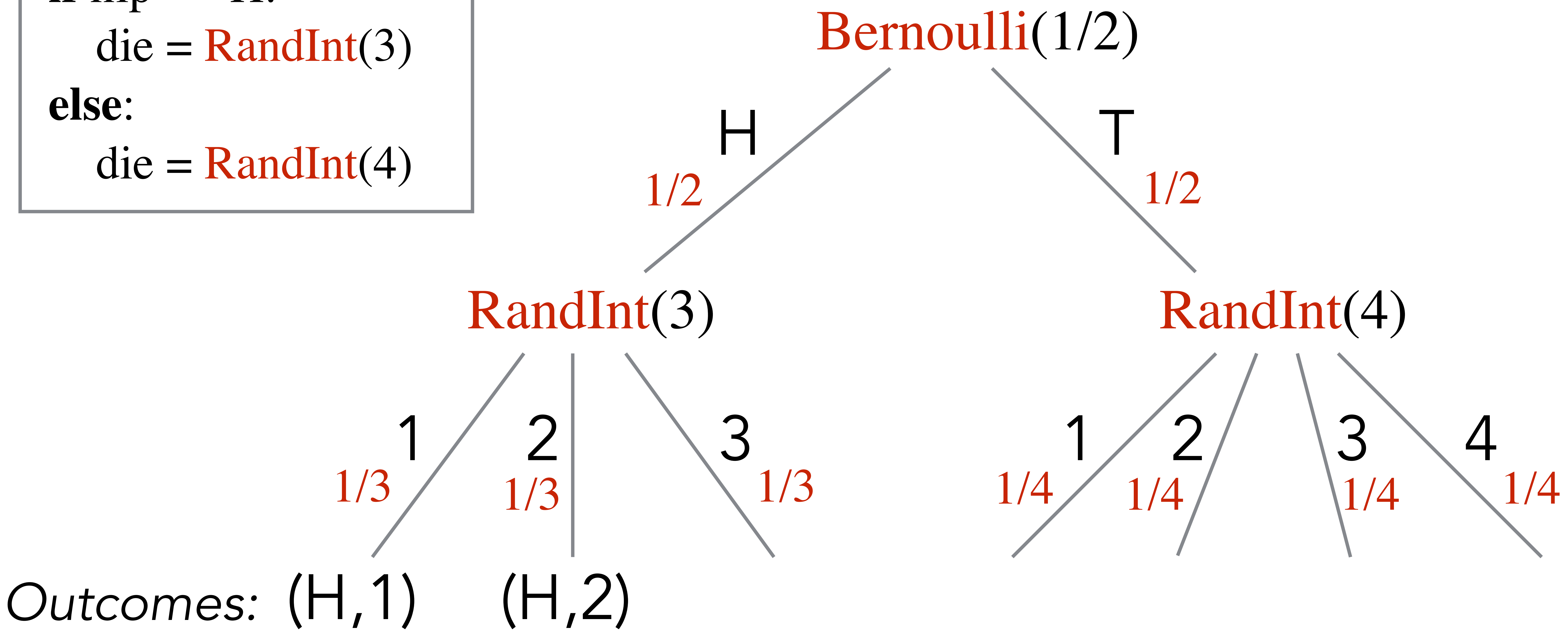
flip = Bernoulli(1/2)
if flip == H:
    die = RandInt(3)
else:
    die = RandInt(4)



Outcomes: (H,1)

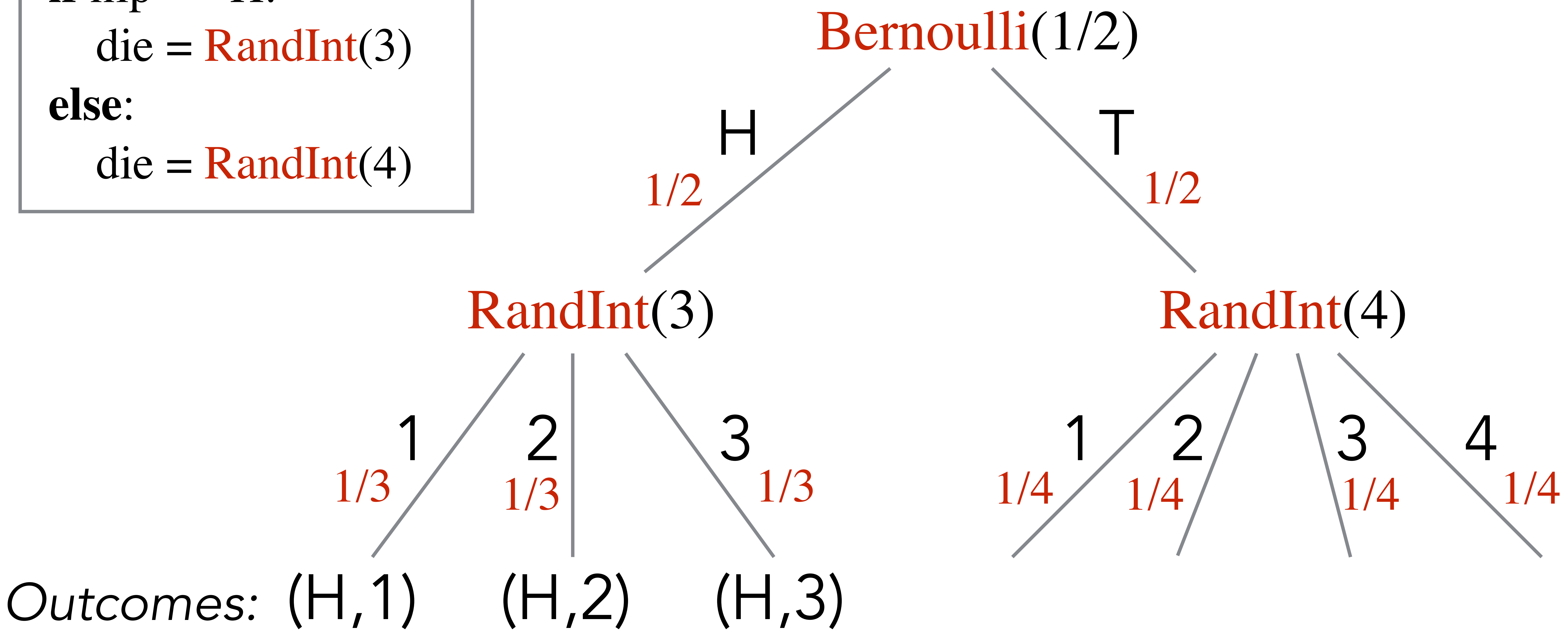
Probability Tree

```
flip = Bernoulli(1/2)
if flip == H:
    die = RandInt(3)
else:
    die = RandInt(4)
```



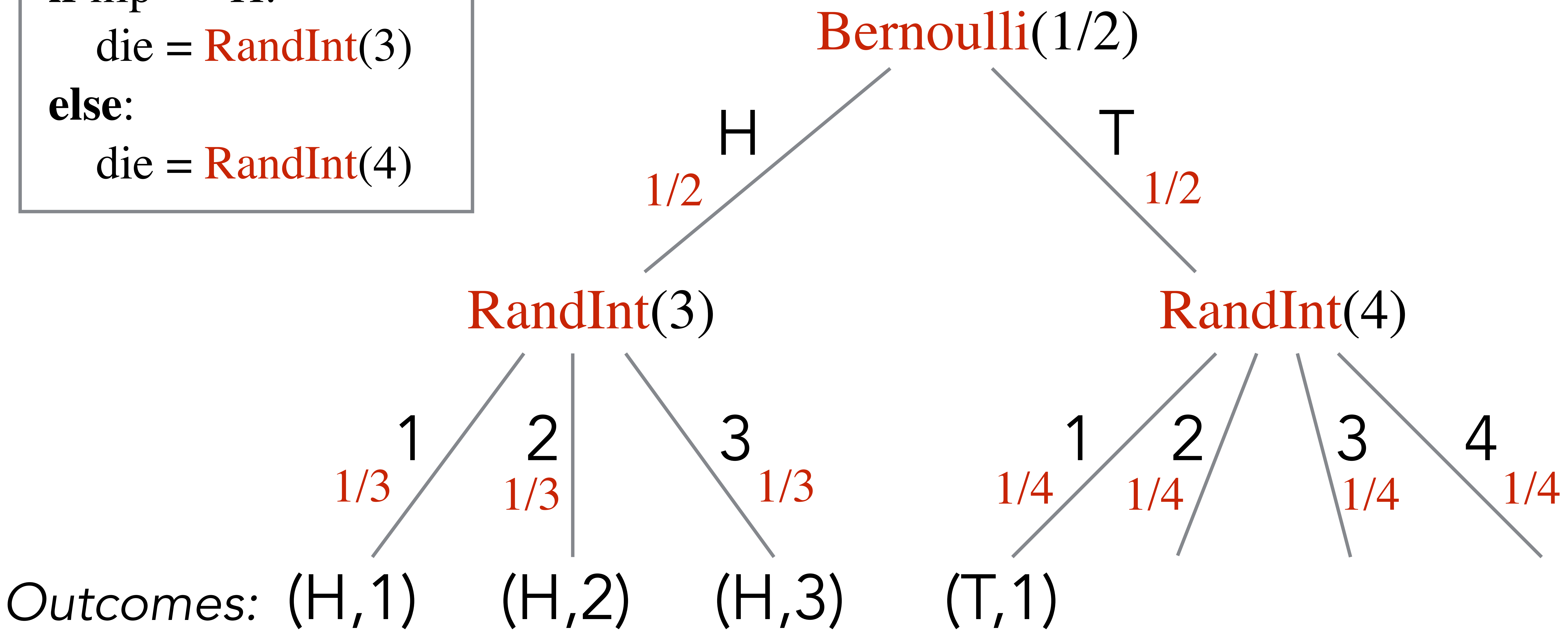
Probability Tree

```
flip = Bernoulli(1/2)
if flip == H:
    die = RandInt(3)
else:
    die = RandInt(4)
```



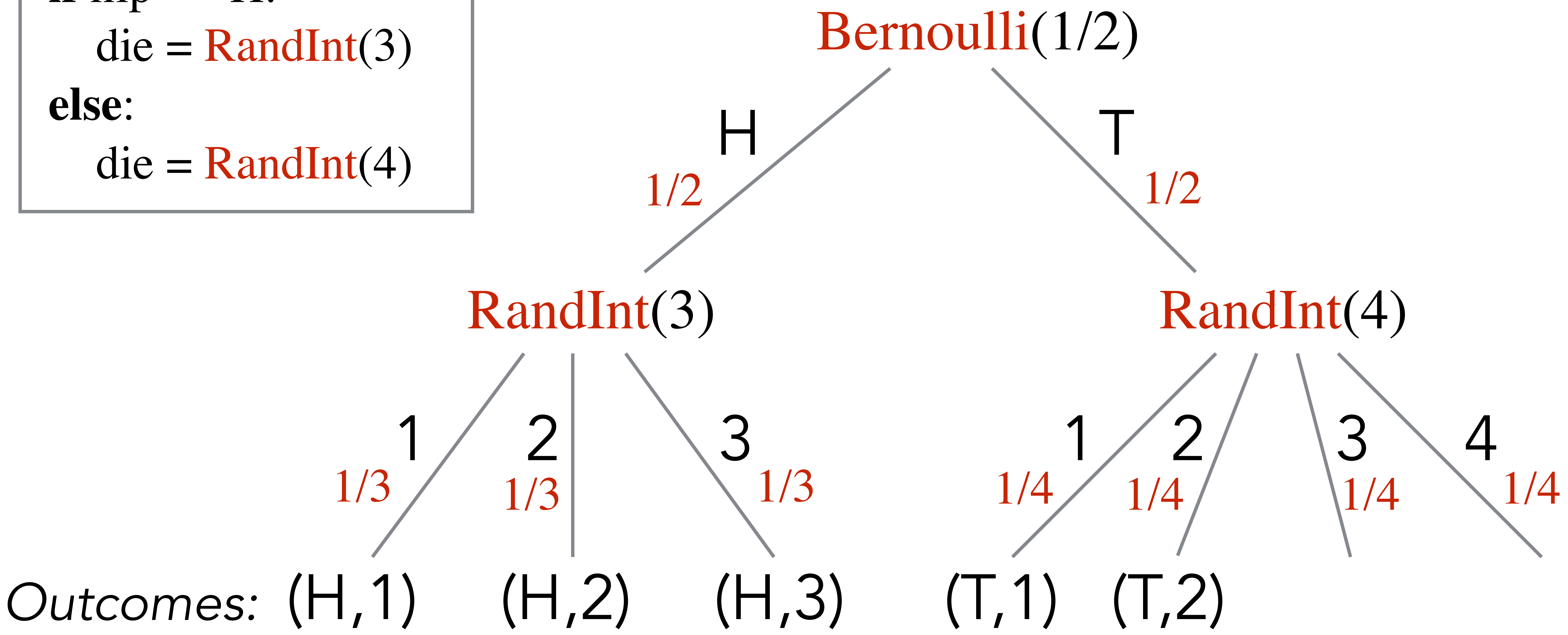
Probability Tree

```
flip = Bernoulli(1/2)
if flip == H:
    die = RandInt(3)
else:
    die = RandInt(4)
```



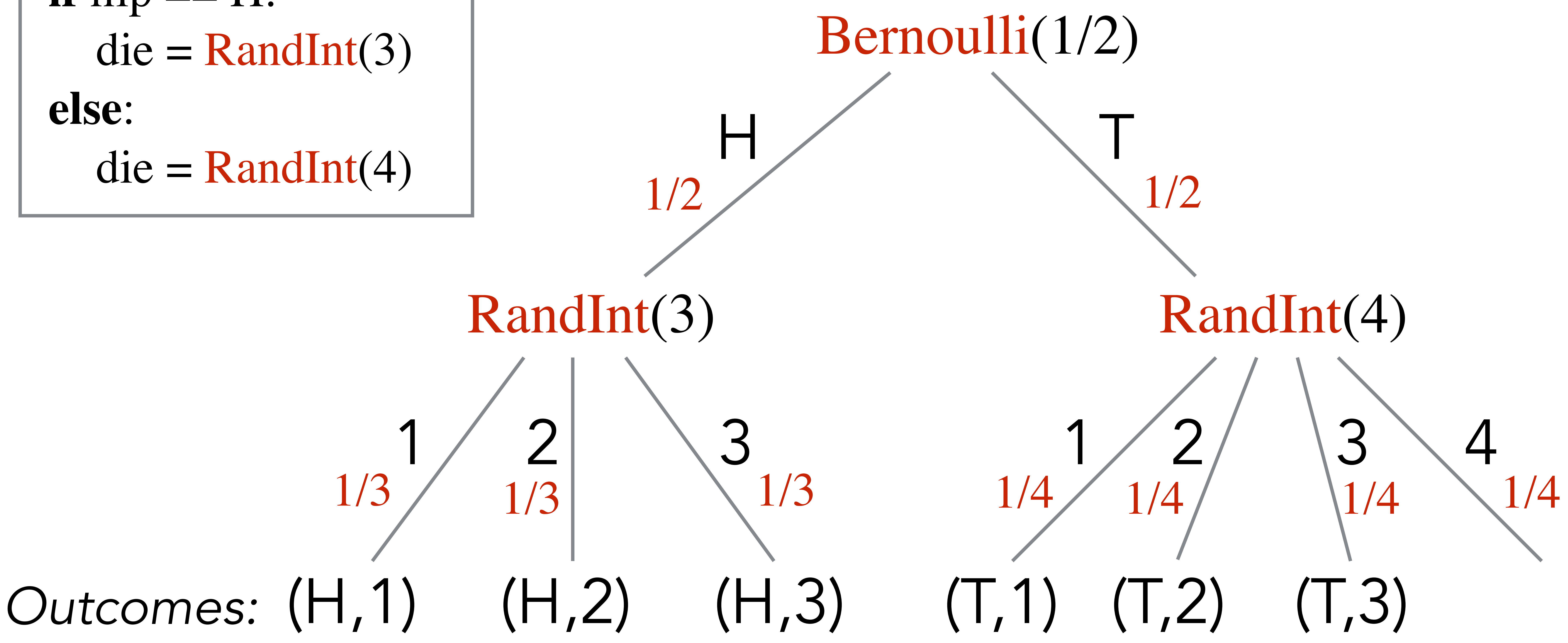
Probability Tree

```
flip = Bernoulli(1/2)
if flip == H:
    die = RandInt(3)
else:
    die = RandInt(4)
```



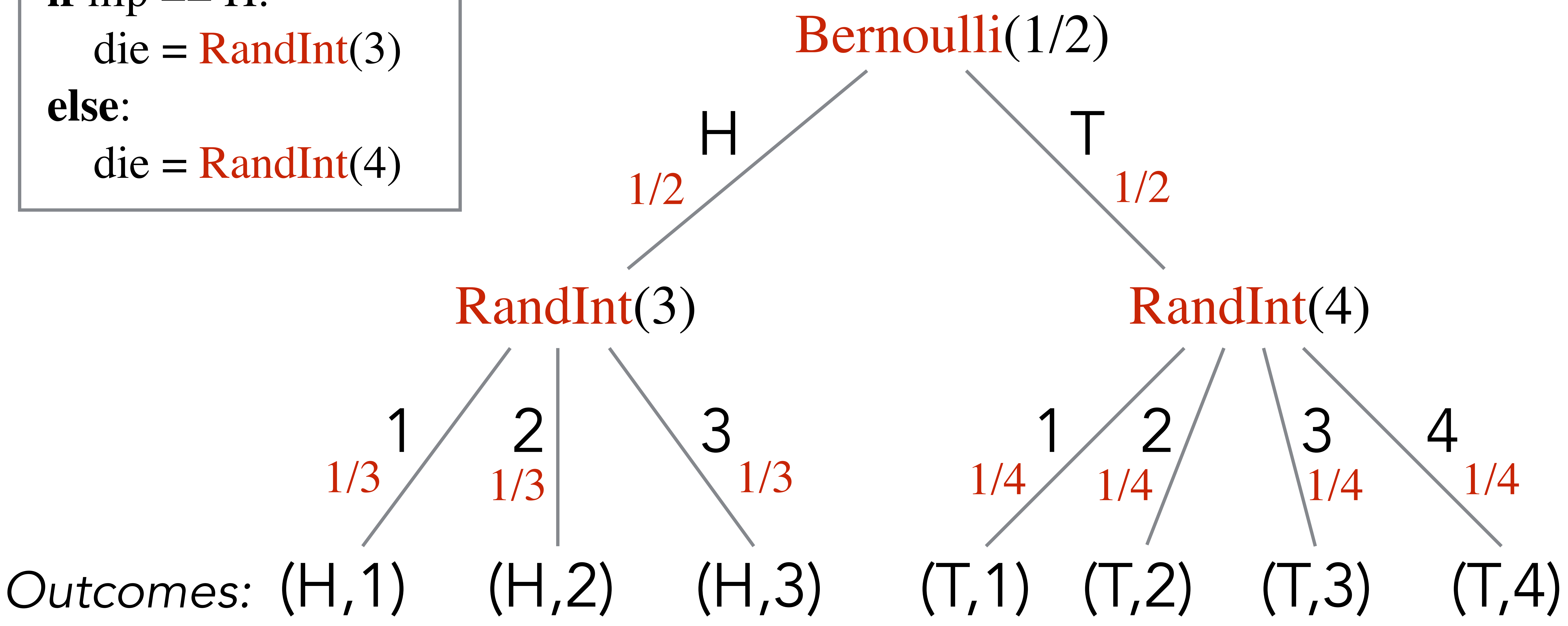
Probability Tree

```
flip = Bernoulli(1/2)
if flip == H:
    die = RandInt(3)
else:
    die = RandInt(4)
```



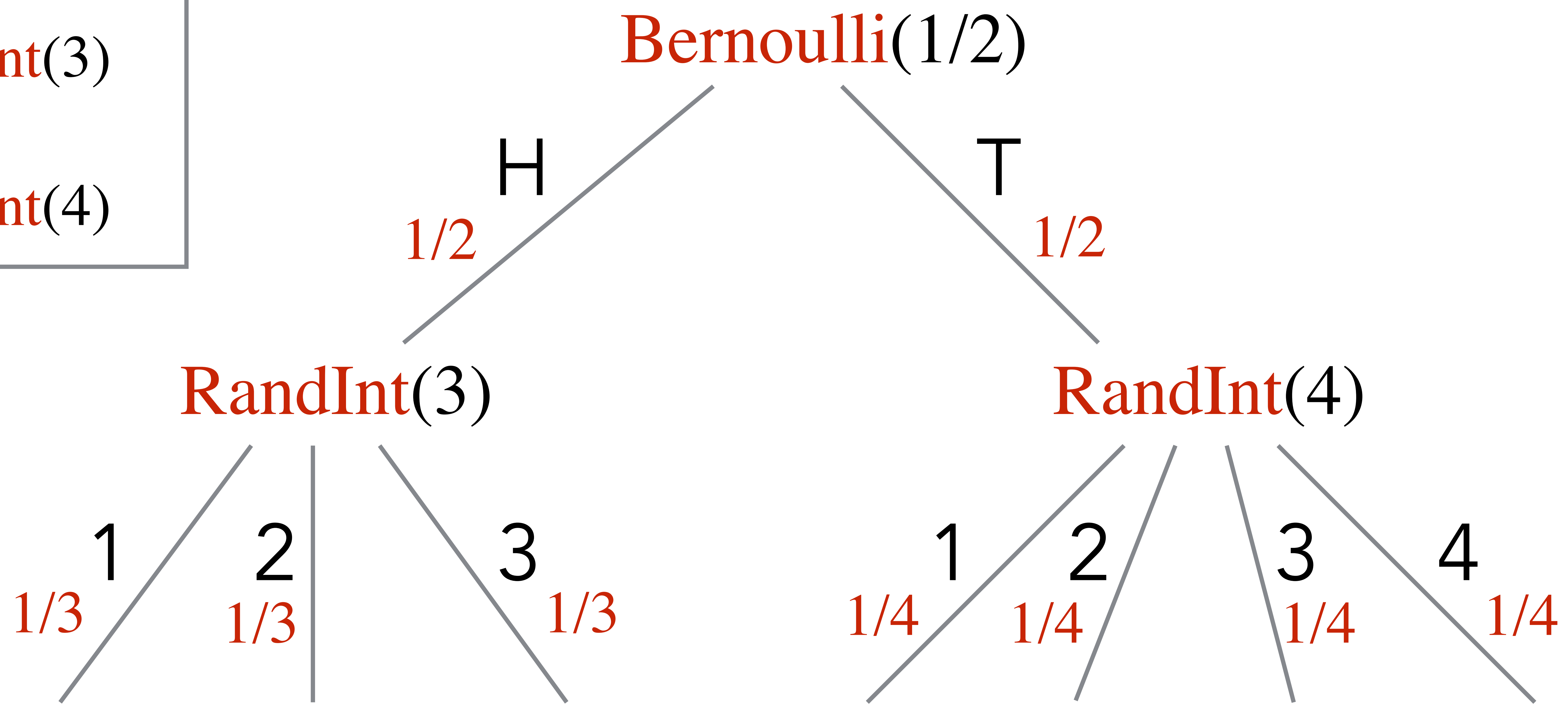
Probability Tree

```
flip = Bernoulli(1/2)
if flip == H:
    die = RandInt(3)
else:
    die = RandInt(4)
```



Probability Tree

```
flip = Bernoulli(1/2)
if flip == H:
    die = RandInt(3)
else:
    die = RandInt(4)
```

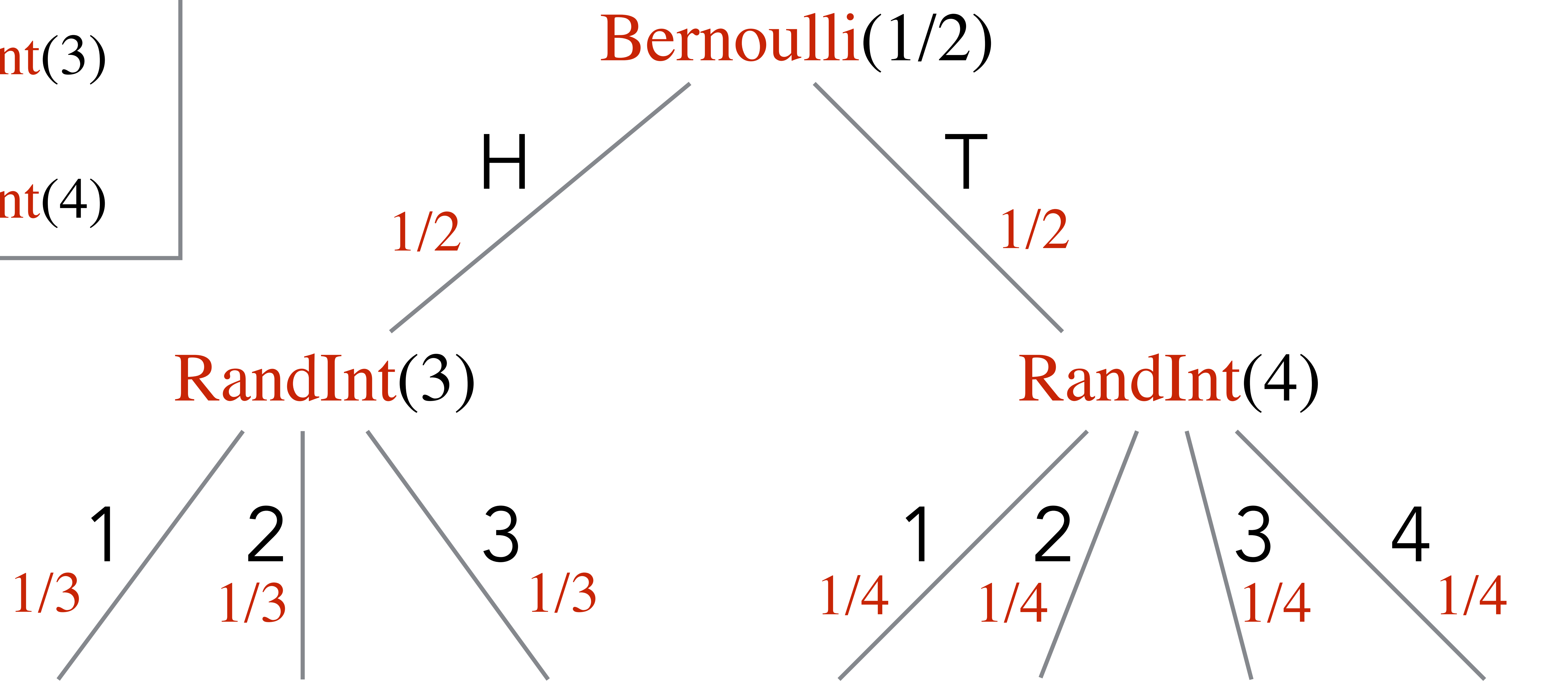


Outcomes: (H,1) (H,2) (H,3) (T,1) (T,2) (T,3) (T,4)

Prob:

Probability Tree

```
flip = Bernoulli(1/2)
if flip == H:
    die = RandInt(3)
else:
    die = RandInt(4)
```

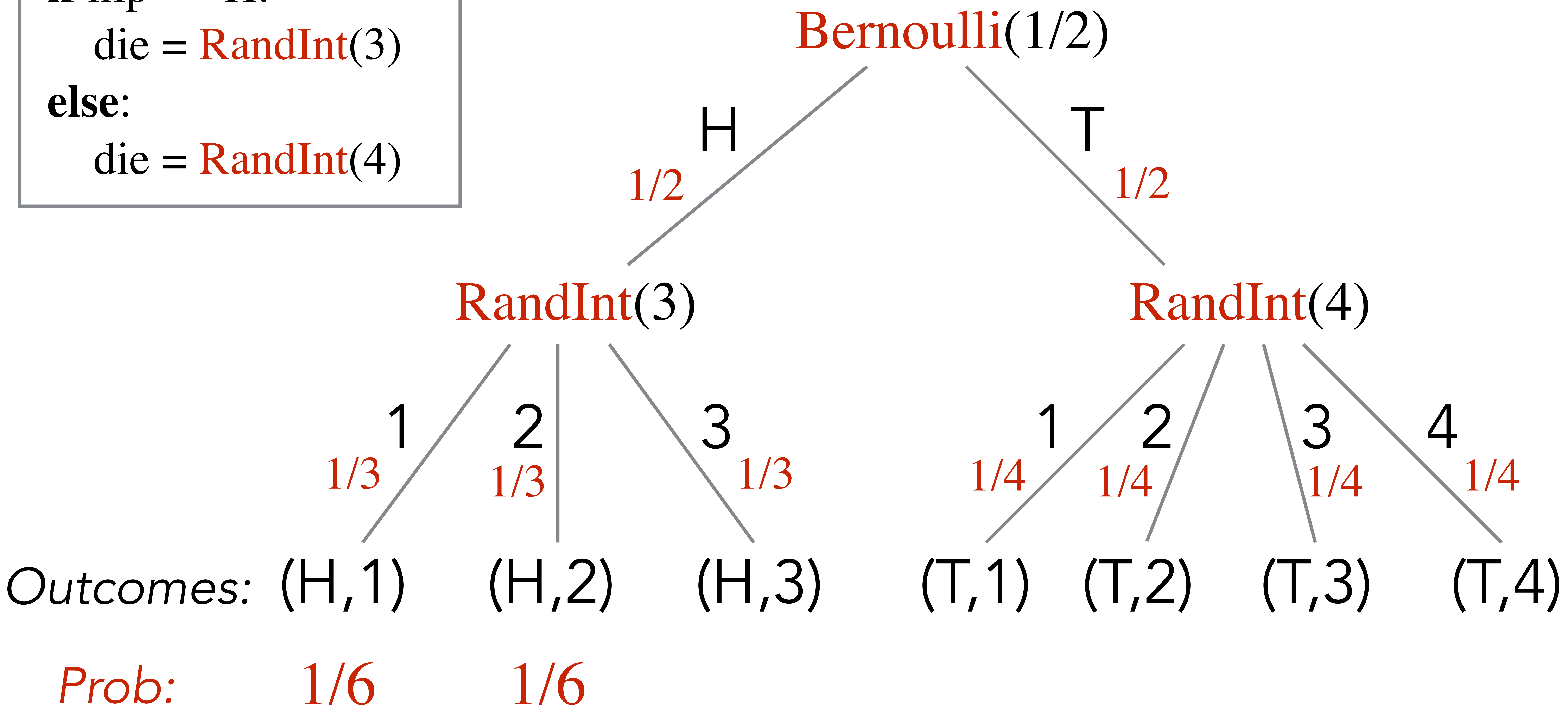


Outcomes: (H,1) (H,2) (H,3) (T,1) (T,2) (T,3) (T,4)

Prob: 1/6

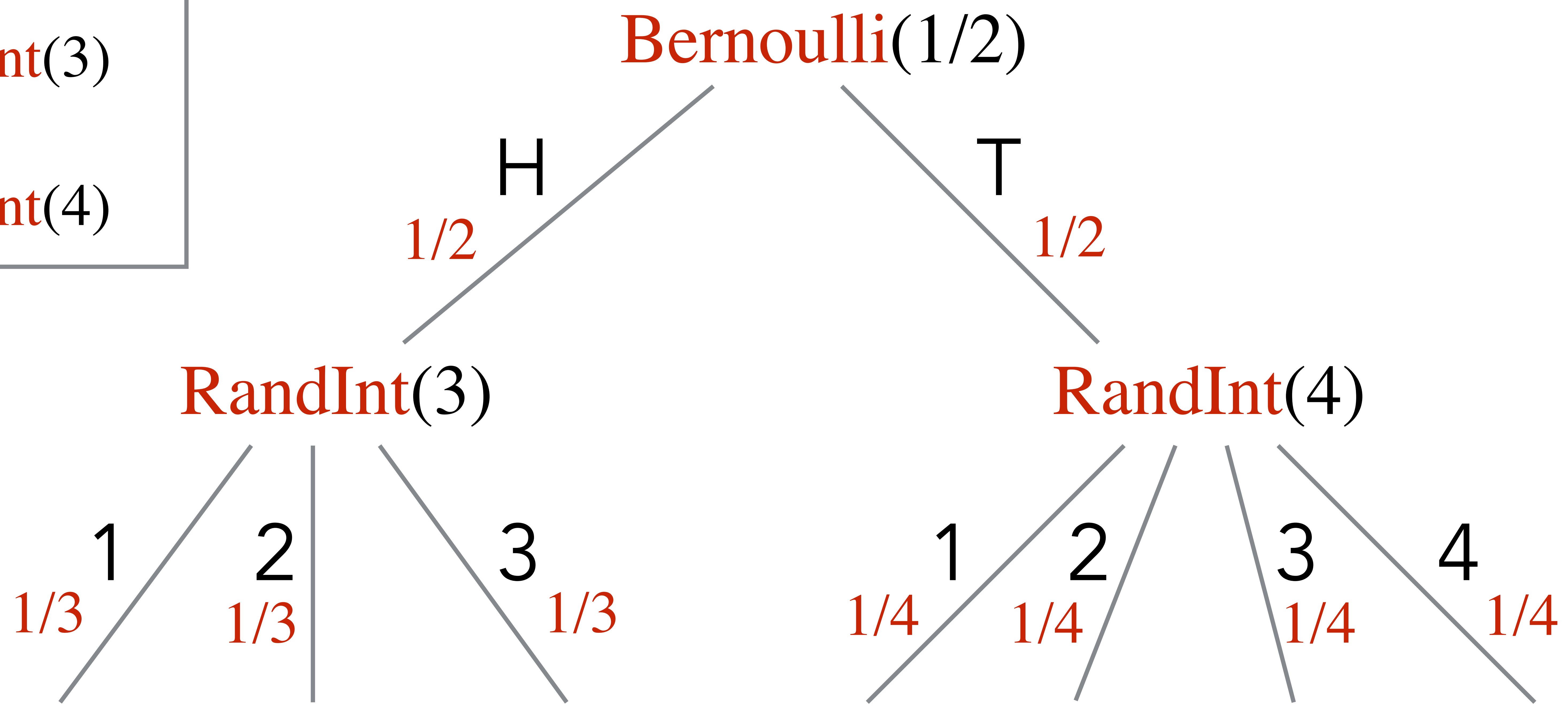
Probability Tree

```
flip = Bernoulli(1/2)
if flip == H:
    die = RandInt(3)
else:
    die = RandInt(4)
```



Probability Tree

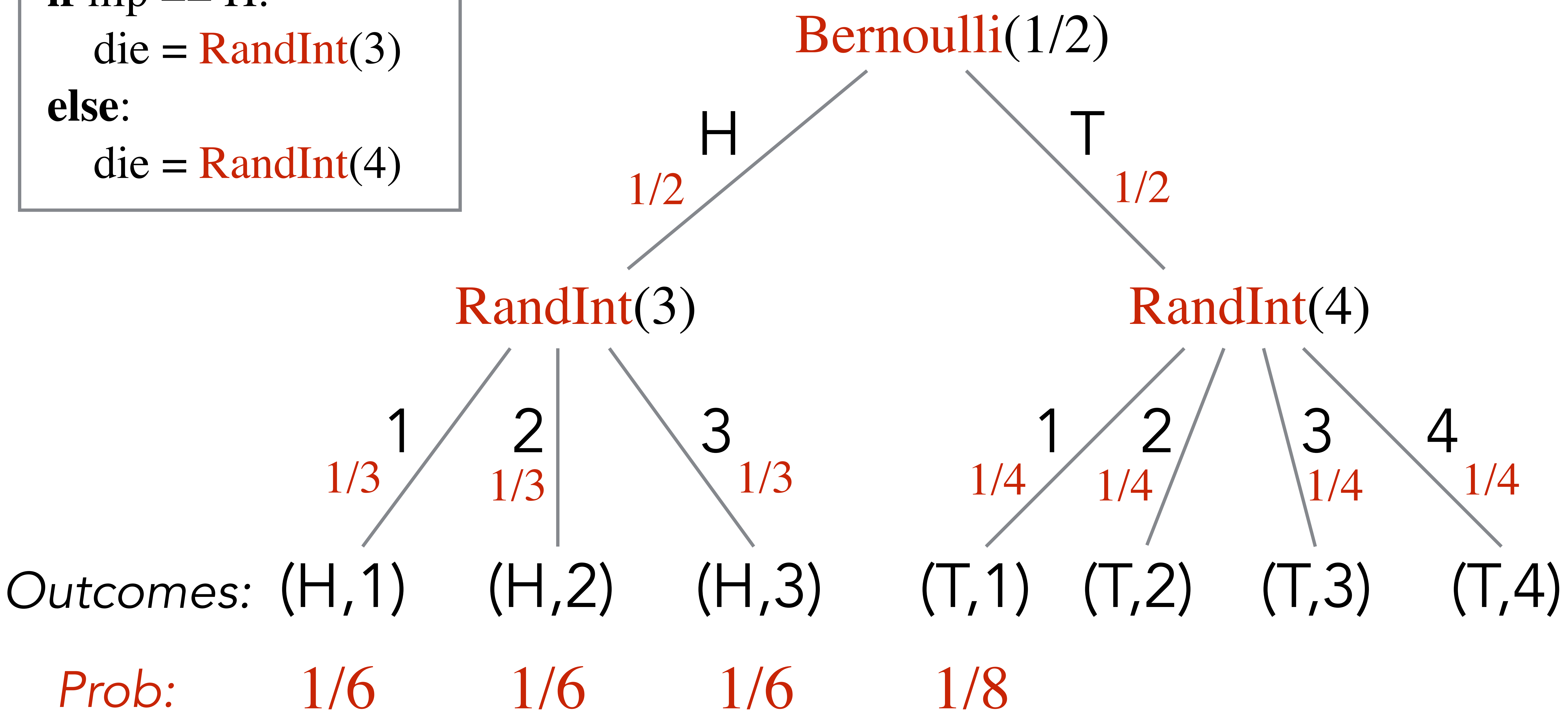
```
flip = Bernoulli(1/2)
if flip == H:
    die = RandInt(3)
else:
    die = RandInt(4)
```



Outcomes:	(H,1)	(H,2)	(H,3)	(T,1)	(T,2)	(T,3)	(T,4)
Prob:	1/6	1/6	1/6				

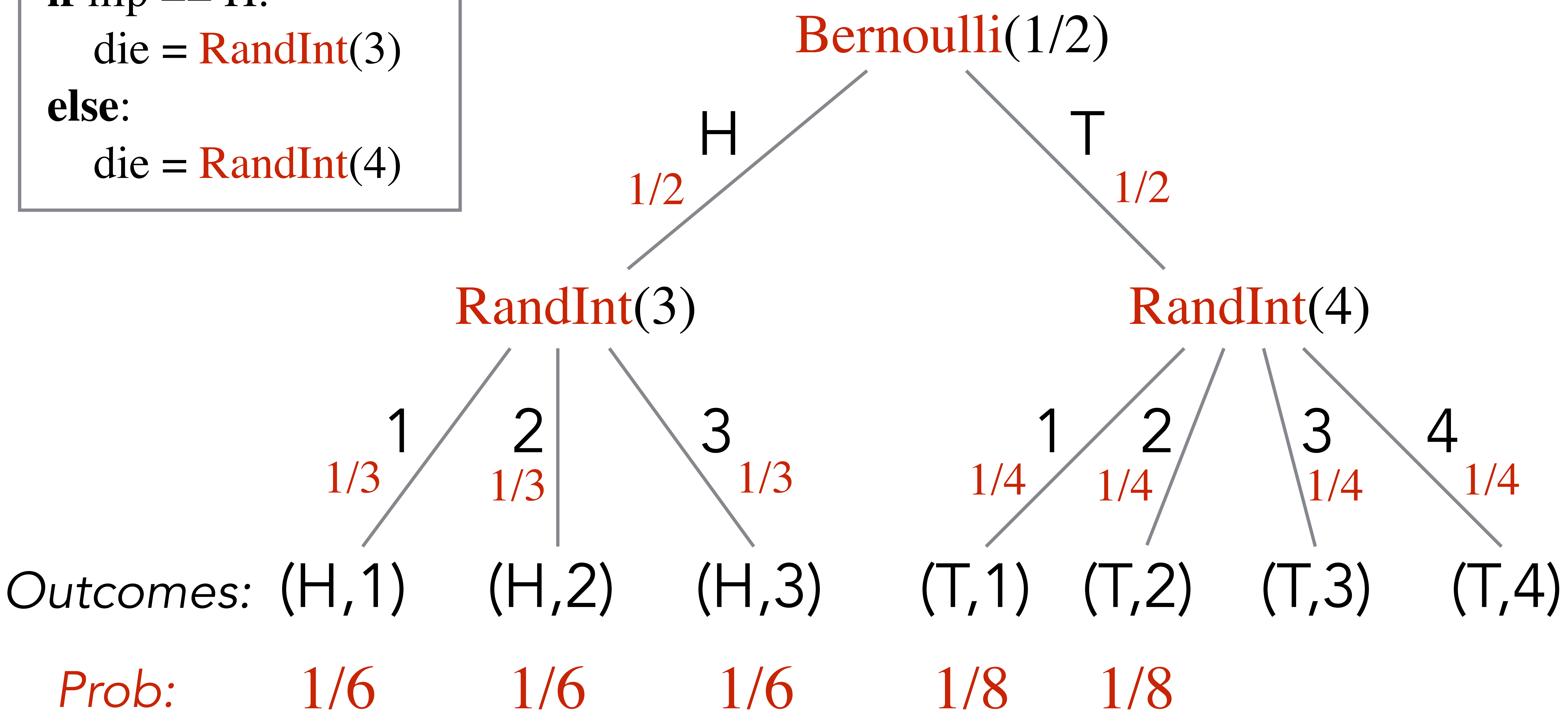
Probability Tree

```
flip = Bernoulli(1/2)
if flip == H:
    die = RandInt(3)
else:
    die = RandInt(4)
```



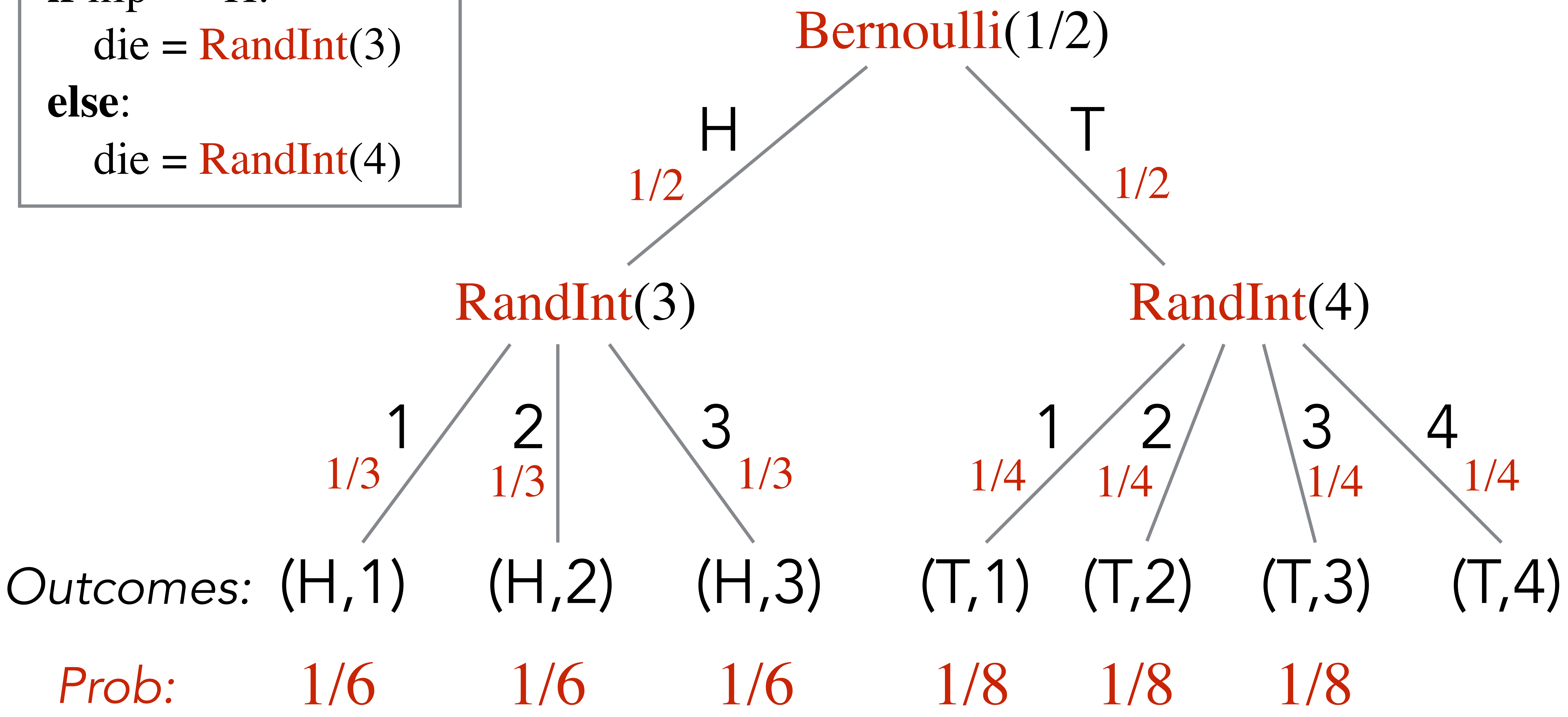
Probability Tree

```
flip = Bernoulli(1/2)
if flip == H:
    die = RandInt(3)
else:
    die = RandInt(4)
```



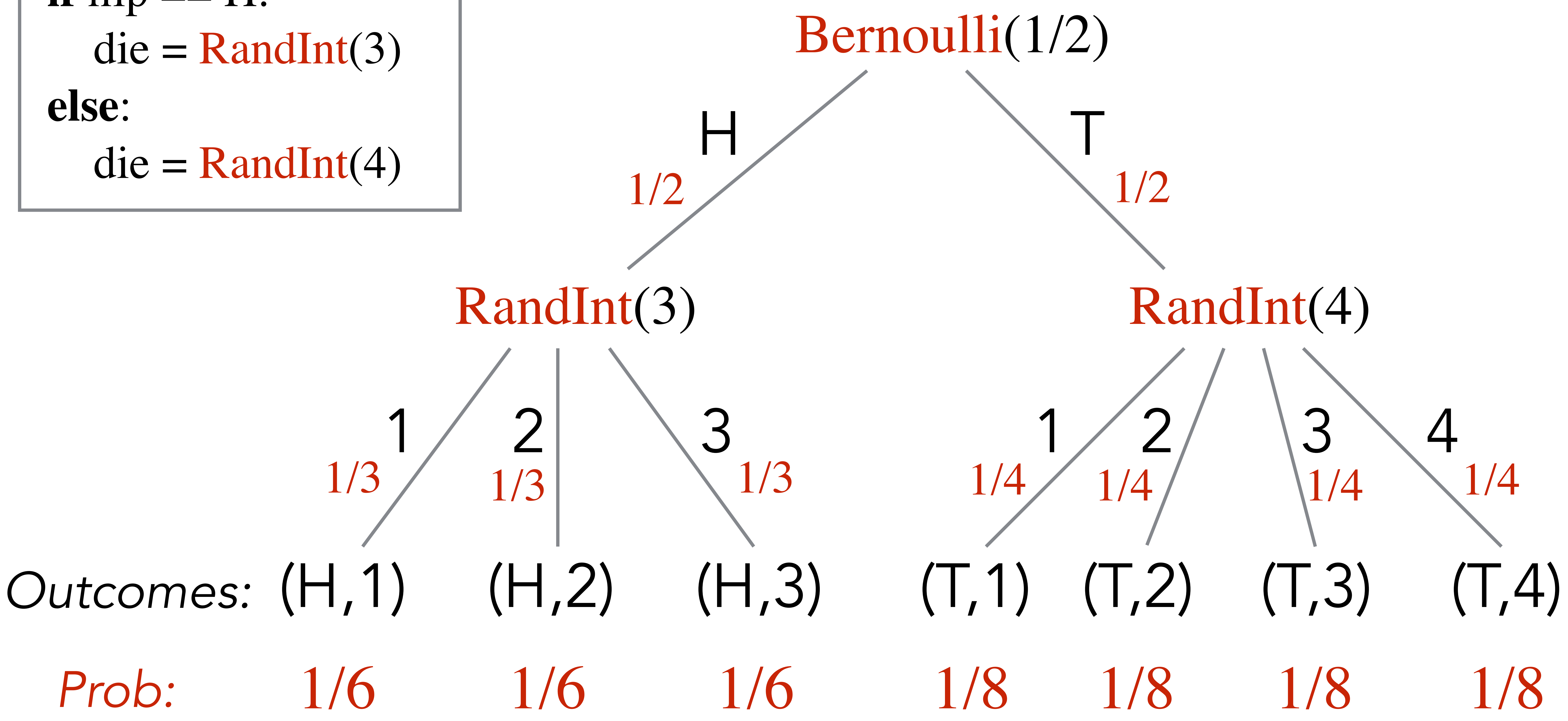
Probability Tree

```
flip = Bernoulli(1/2)
if flip == H:
    die = RandInt(3)
else:
    die = RandInt(4)
```



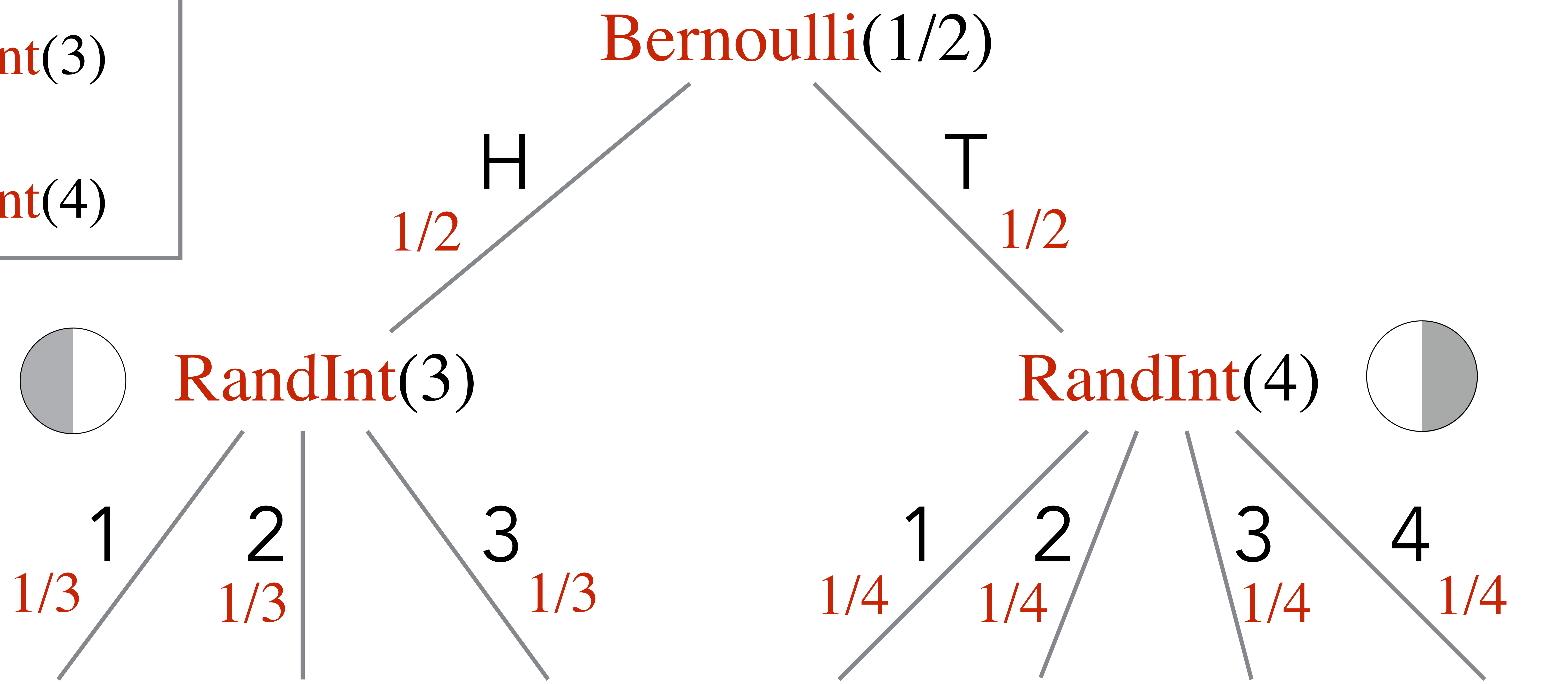
Probability Tree

```
flip = Bernoulli(1/2)
if flip == H:
    die = RandInt(3)
else:
    die = RandInt(4)
```



Probability Tree

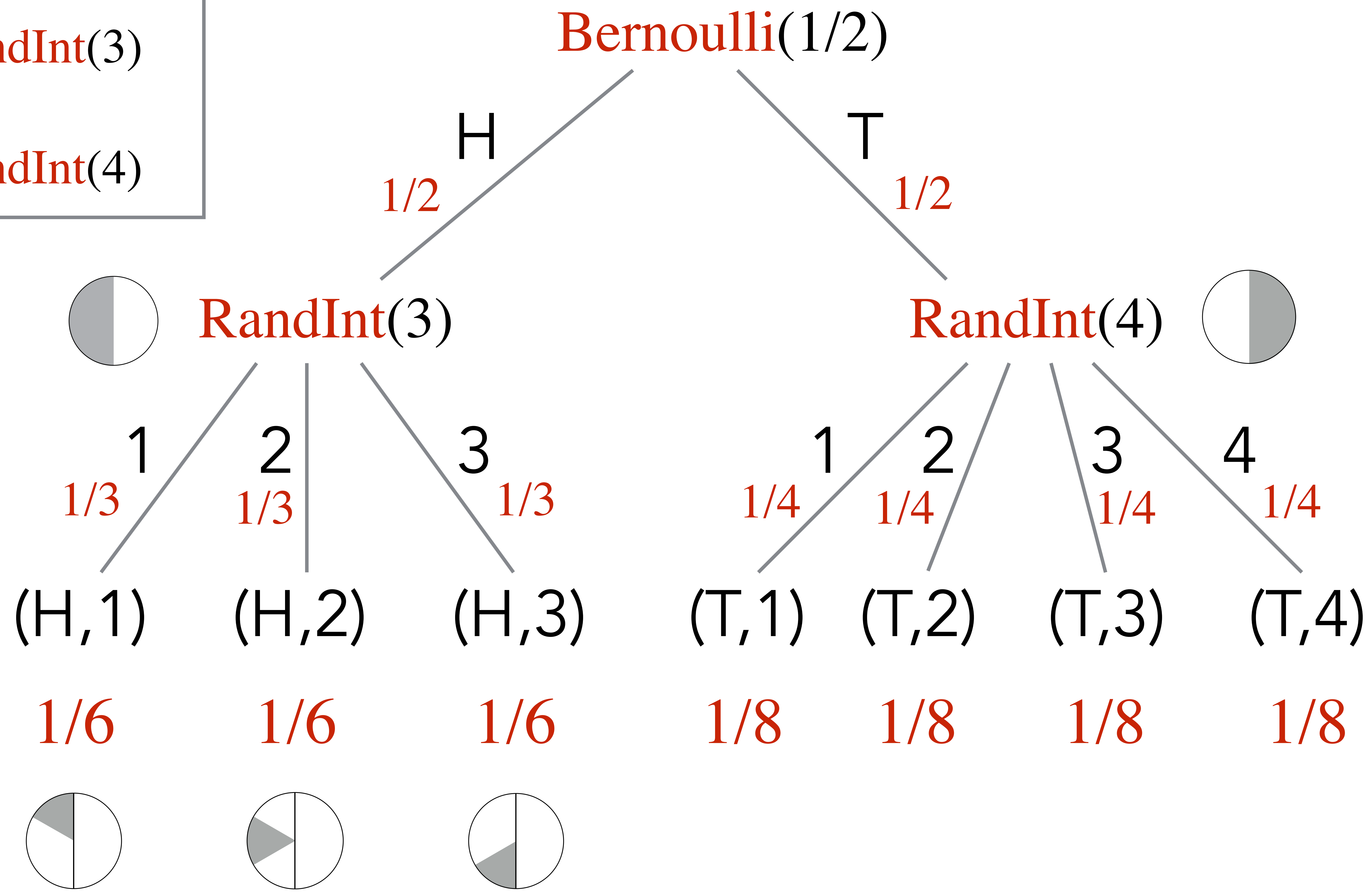
```
flip = Bernoulli(1/2)
if flip == H:
    die = RandInt(3)
else:
    die = RandInt(4)
```



Outcomes:	(H,1)	(H,2)	(H,3)	(T,1)	(T,2)	(T,3)	(T,4)
Prob:	1/6	1/6	1/6	1/8	1/8	1/8	1/8

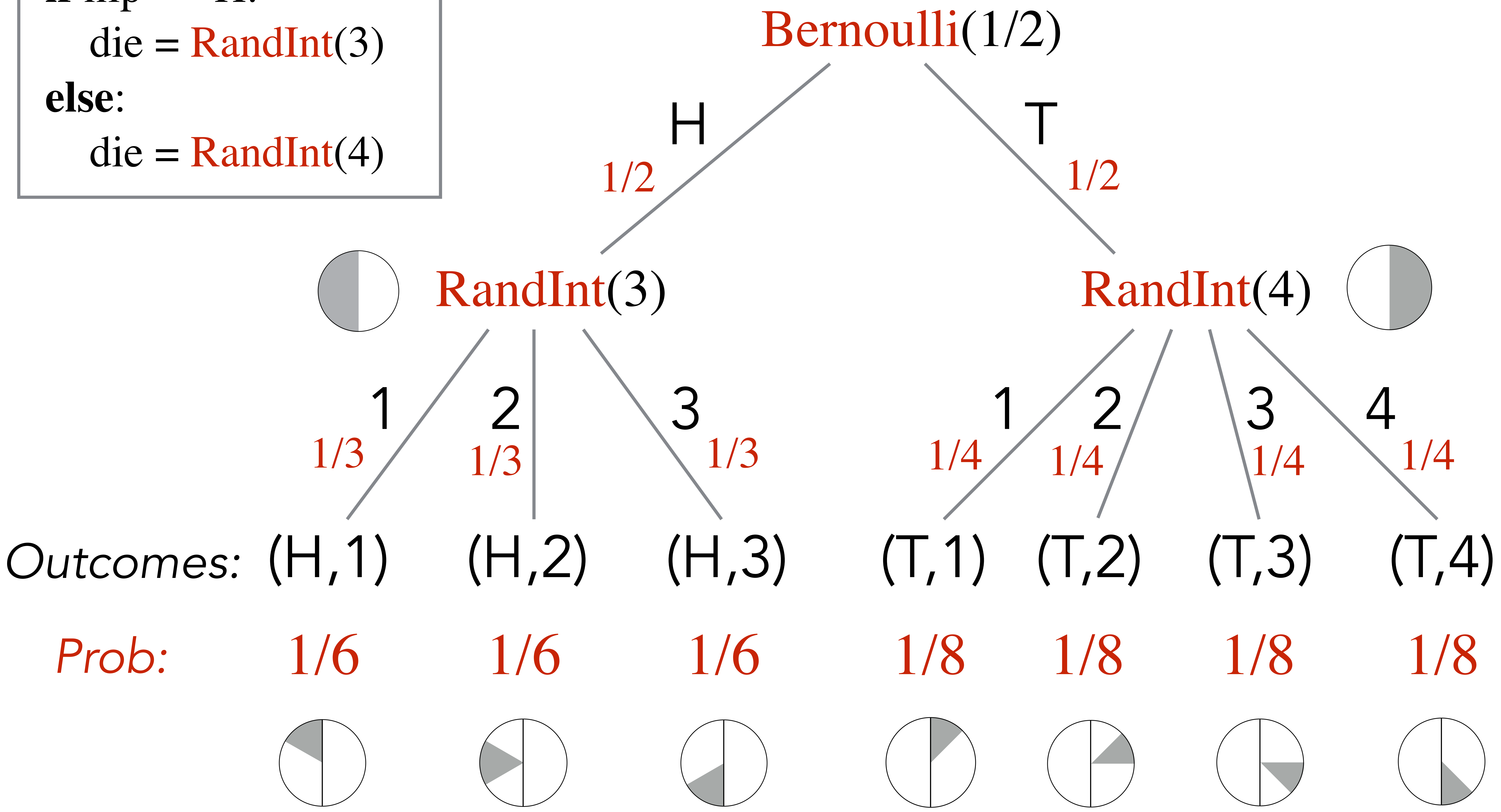
Probability Tree

```
flip = Bernoulli(1/2)
if flip == H:
    die = RandInt(3)
else:
    die = RandInt(4)
```

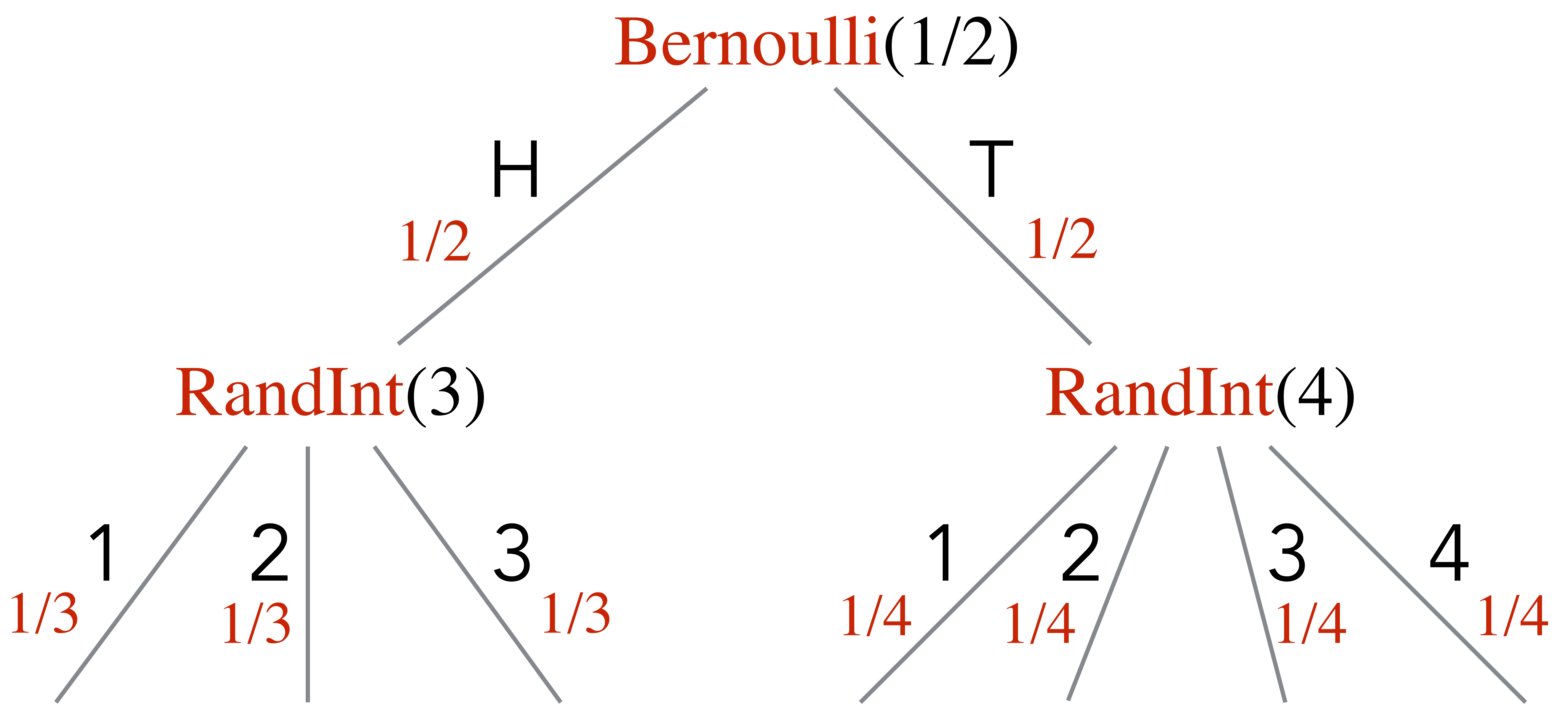


Probability Tree

```
flip = Bernoulli(1/2)
if flip == H:
    die = RandInt(3)
else:
    die = RandInt(4)
```

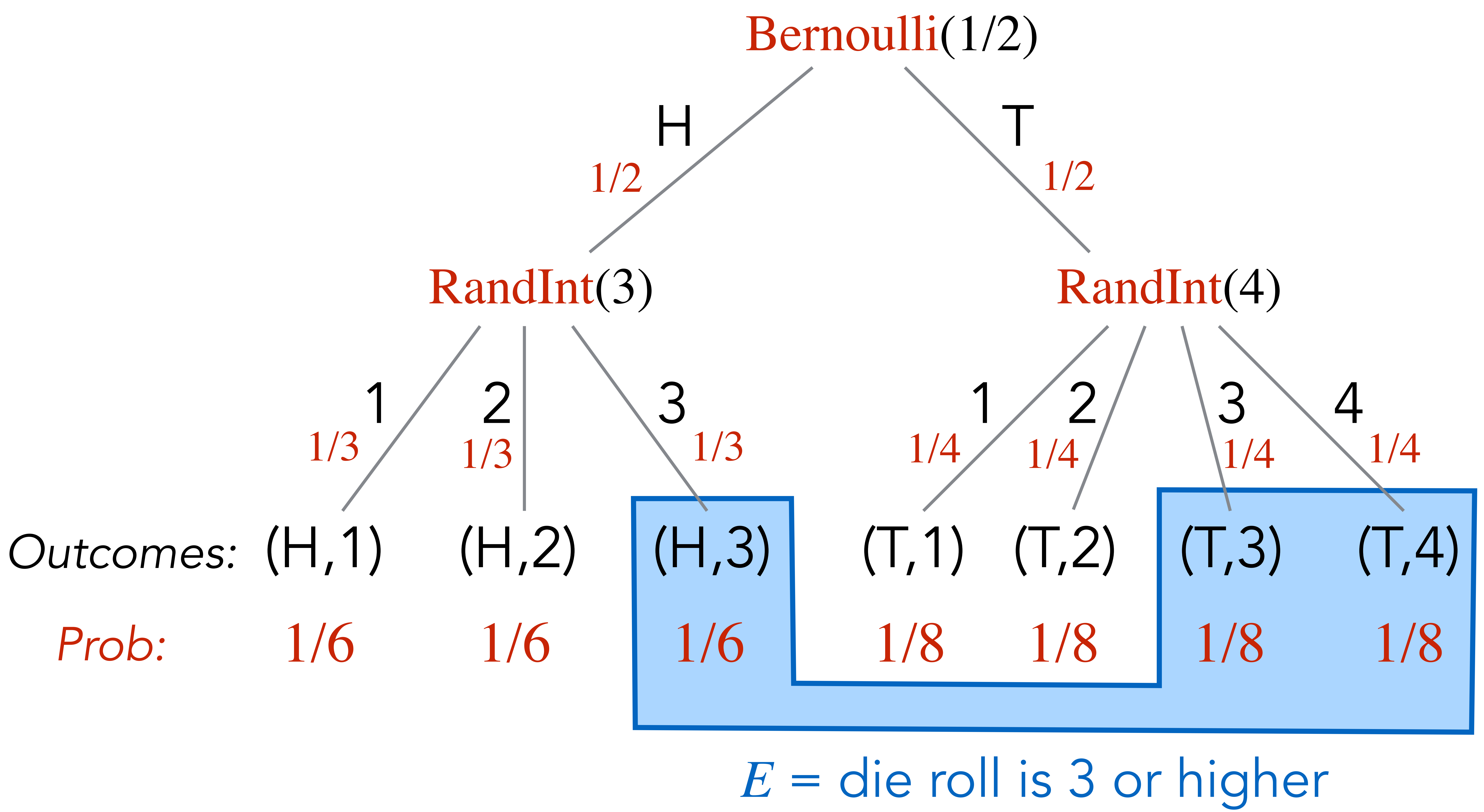


Events

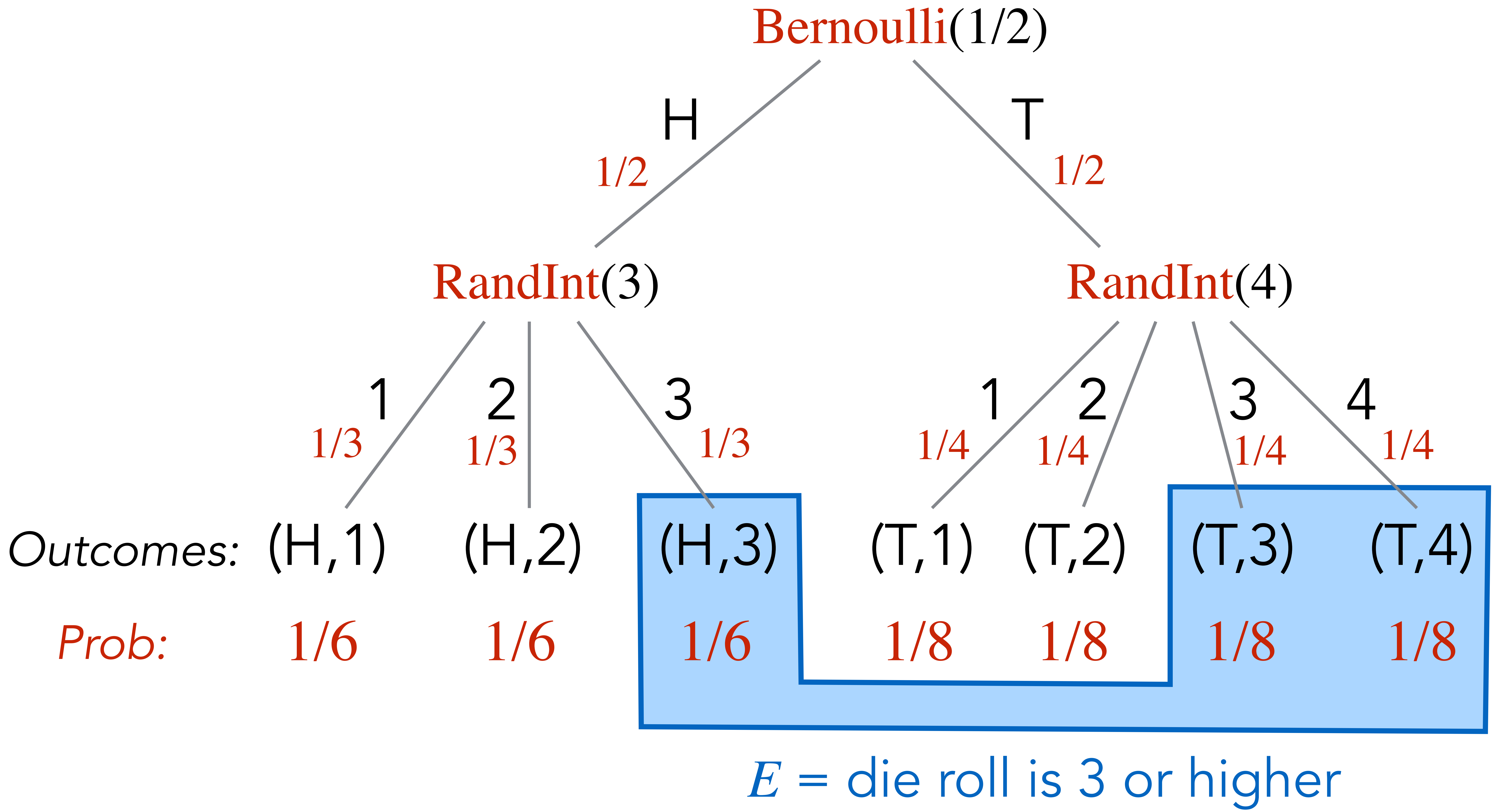


Outcomes:	(H,1)	(H,2)	(H,3)	(T,1)	(T,2)	(T,3)	(T,4)
Prob:	1/6	1/6	1/6	1/8	1/8	1/8	1/8

Events

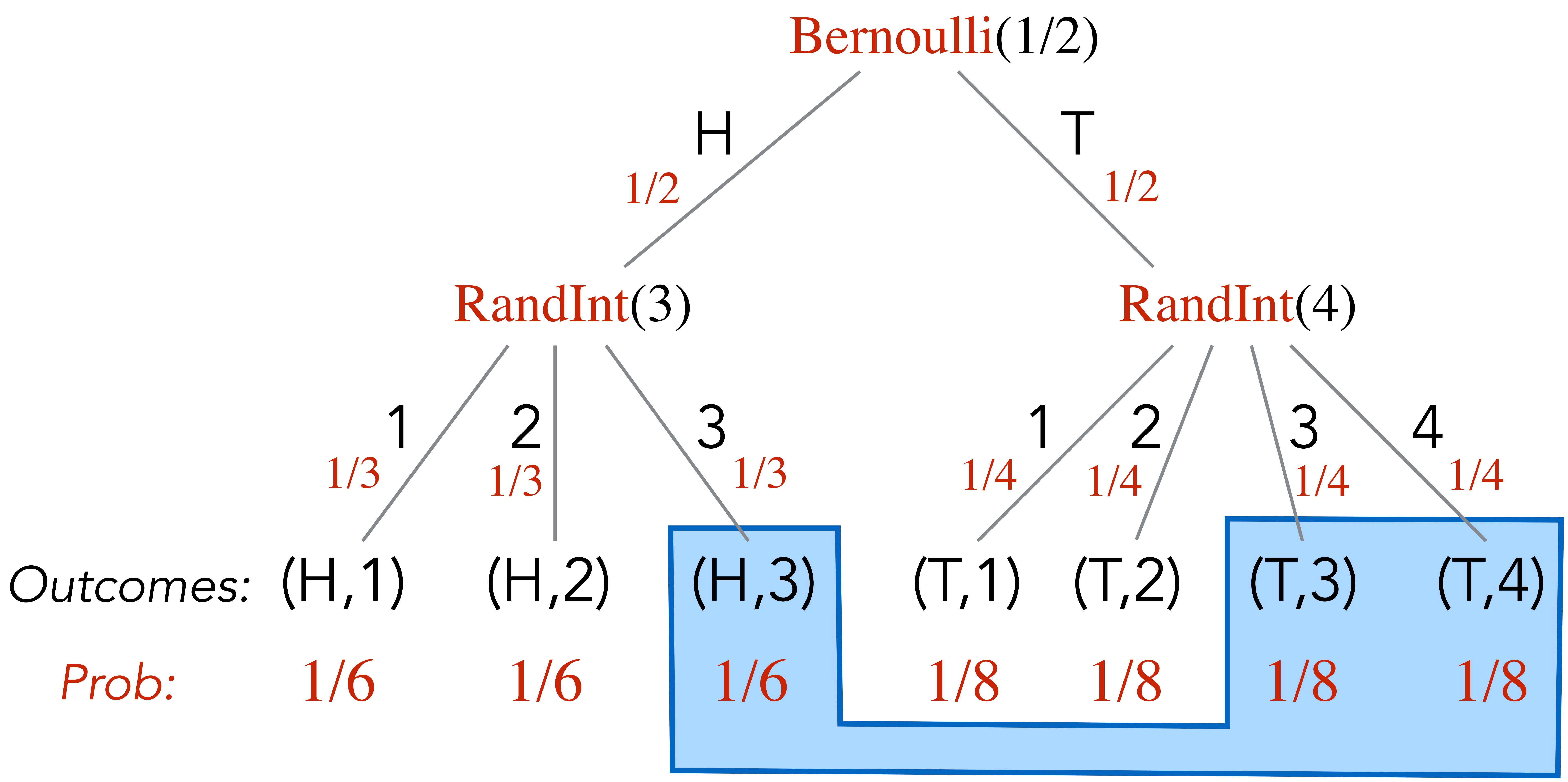

Bernoulli(1/2)
H
1/2
T
1/2
RandInt(3)
RandInt(4)
1
1/3
2
1/3
3
1/3
1
1/4
2
1/4
3
1/4
4
1/4
Outcomes: (H,1) (H,2) (H,3) (T,1) (T,2) (T,3) (T,4)
Prob: 1/6 1/6 1/6 1/8 1/8 1/8 1/8
E = die roll is 3 or higher

Events



$$\Pr[E] = 1/6 + 1/8 + 1/8 = 5/12$$

Events



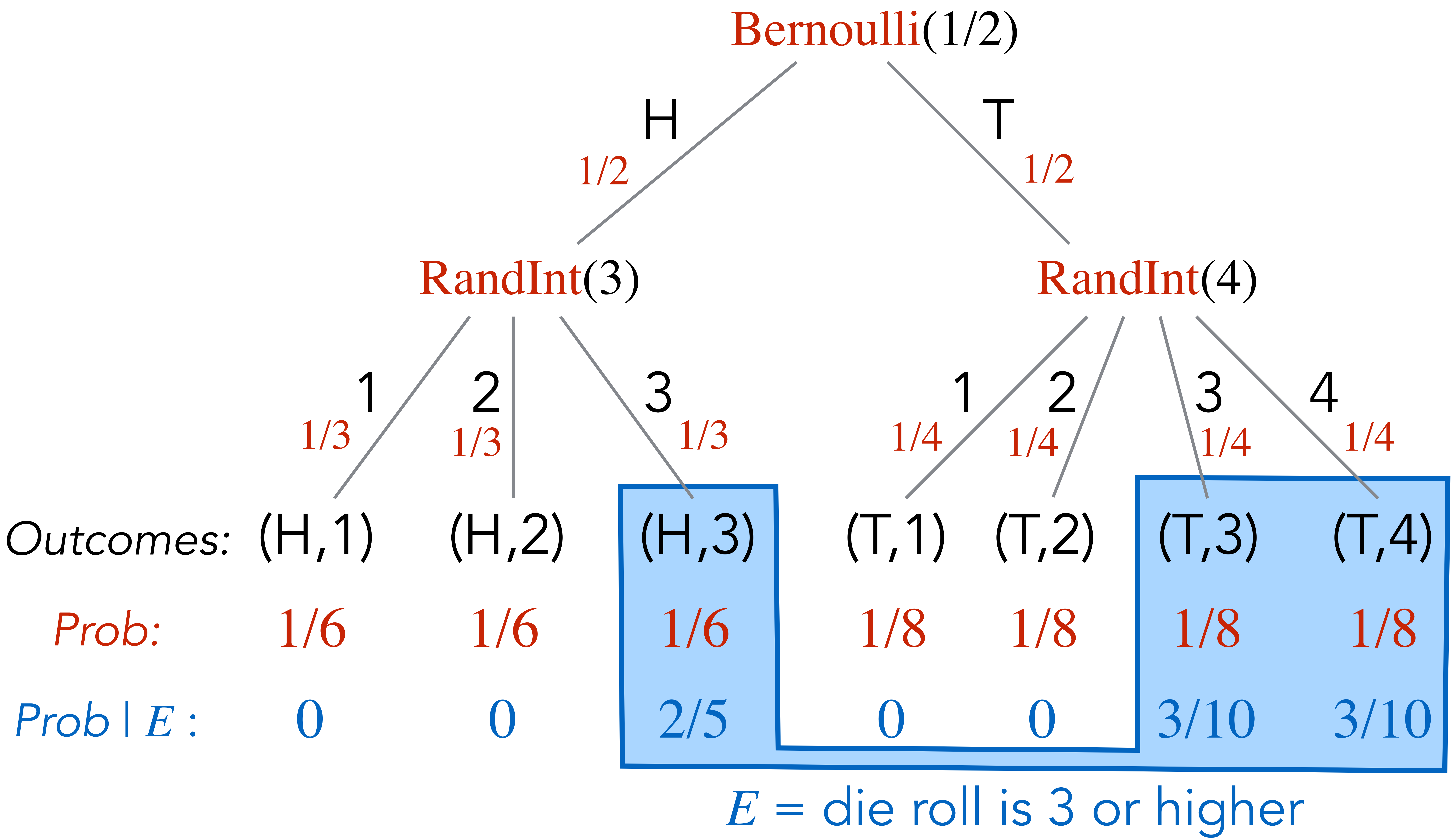
Extend Pr to:

$$\Pr : \mathcal{P}(\Omega) \to [0,1]$$

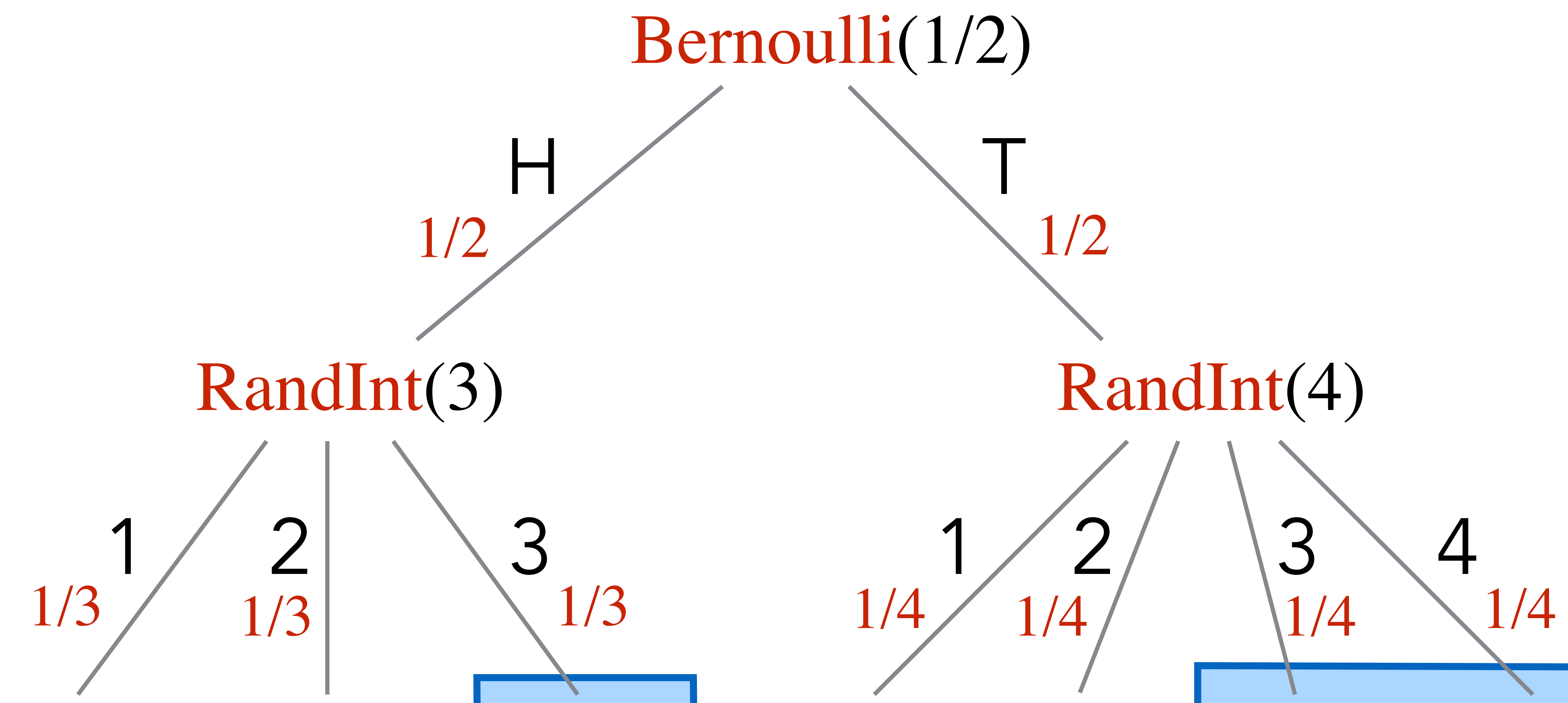
E = die roll is 3 or higher

$$\Pr[E] = 1/6 + 1/8 + 1/8 = 5/12$$

Conditional Probability



Conditional Probability

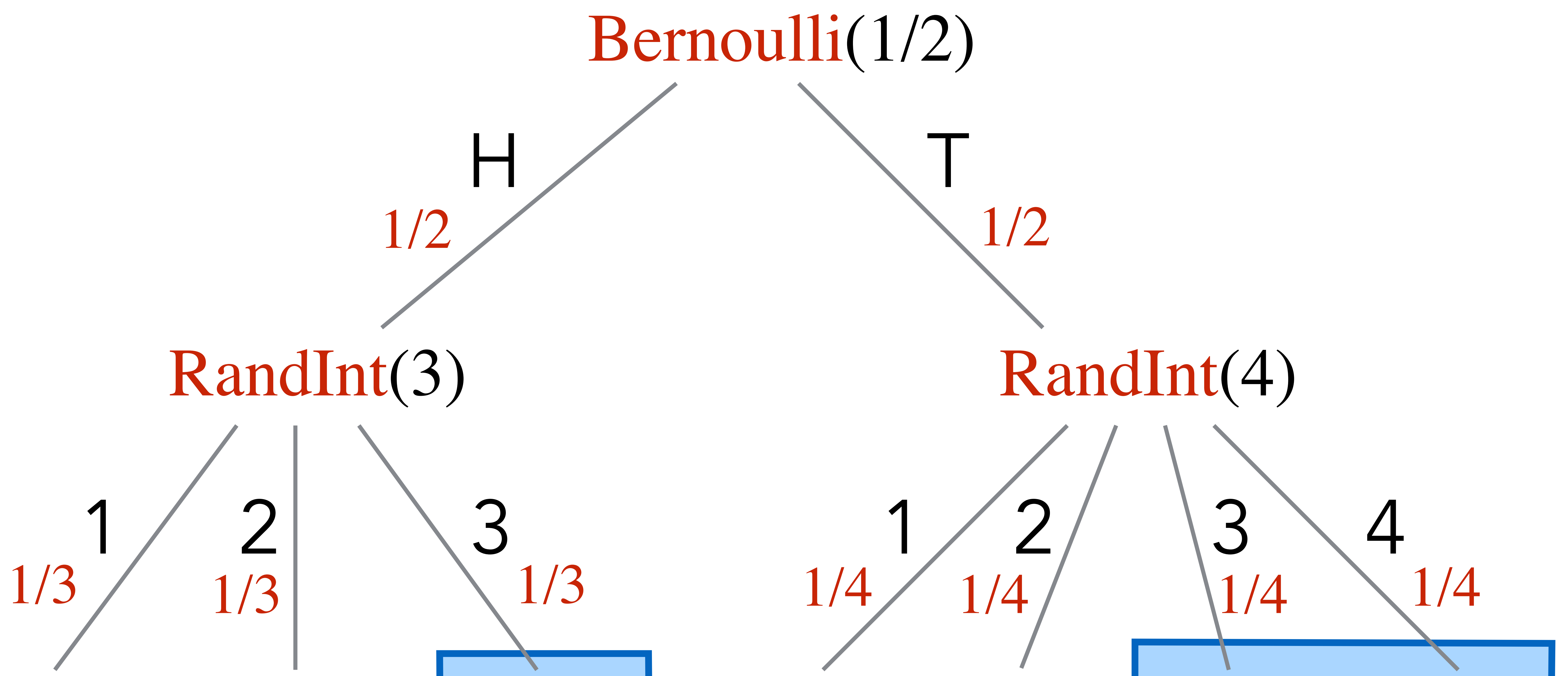


Outcomes:	(H,1)	(H,2)	(H,3)	(T,1)	(T,2)	(T,3)	(T,4)
Prob:	1/6	1/6	1/6	1/8	1/8	1/8	1/8
Prob | E :	0	0	2/5	0	0	3/10	3/10

E = die roll is 3 or higher

A = Tails was flipped

Conditional Probability



Outcomes:	(H,1)	(H,2)	(H,3)	(T,1)	(T,2)	(T,3)	(T,4)
Prob:	1/6	1/6	1/6	1/8	1/8	1/8	1/8
Prob | E :	0	0	2/5	0	0	3/10	3/10

E = die roll is 3 or higher

A = Tails was flipped

$\Pr[A \mid E] = 3/5$

Conditional Probability –> Chain Rule

Conditional Probability –> Chain Rule

$$\Pr[A \cap B] = \Pr[A] \cdot \Pr[B \mid A]$$

Conditional Probability –> Chain Rule

$$\Pr[A \cap B] = \Pr[A] \cdot \Pr[B \mid A]$$

Intuition:

Conditional Probability –> Chain Rule

$$\Pr[A \cap B] = \Pr[A] \cdot \Pr[B \mid A]$$

Intuition:

"For A and B to occur:

- first A must occur (probability $\Pr[A]$)
- then B must occur given that A occured (probability $\Pr[B \mid A]$)."

Conditional Probability –> Chain Rule

$$\Pr[A \cap B] = \Pr[A] \cdot \Pr[B \mid A]$$

Intuition:

"For A and B to occur:

- first A must occur (probability $\Pr[A]$)
- then B must occur given that A occured (probability $\Pr[B \mid A]$)."

Generalizes to more than two events.

$$\Pr[A \cap B \cap C] = \Pr[A] \cdot \Pr[B \mid A] \cdot \Pr[C \mid A \cap B]$$

Independence

Independence

Two events A and B are **independent** if

$$\Pr[A \mid B] = \Pr[A].$$

Independence

Two events A and B are **independent** if

$$\Pr[A \mid B] = \Pr[A].$$

This is equivalent to:

$$\Pr[B \mid A] = \Pr[B].$$

Independence

Two events A and B are **independent** if

$$\Pr[A \mid B] = \Pr[A].$$

This is equivalent to:

$$\Pr[B \mid A] = \Pr[B].$$

This is equivalent to:

$$\Pr[A \cap B] = \Pr[A] \cdot \Pr[B]$$

Independence

Two events A and B are **independent** if

$$\Pr[A \mid B] = \Pr[A].$$

This is equivalent to:

$$\Pr[B \mid A] = \Pr[B].$$

This is equivalent to:

$$\Pr[A \cap B] = \Pr[A] \cdot \Pr[B]$$

(except that this can be used even when $\Pr[A] = 0$ or $\Pr[B] = 0$.)

SUMMARY SO FAR

SUMMARY SO FAR

Real World → Code → Probability Tree

$\parallel$

Mathematical Model

- set of outcomes Ω
- a prob. associated with each outcome.

SUMMARY SO FAR

Real World → Code →

Probability Tree
$\parallel$
Mathematical Model
- set of outcomes Ω
- a prob. associated with each outcome.

Events

SUMMARY SO FAR

Real World → Code → Probability Tree

=

Mathematical Model

- set of outcomes Ω
- a prob. associated with each outcome.

Events

Conditional probability:

$$\Pr[A \mid B] = \Pr[A \cap B] / \Pr[B]$$

SUMMARY SO FAR

Real World → Code → Probability Tree

$\parallel$

Mathematical Model

- set of outcomes Ω
- a prob. associated with each outcome.

Events

Conditional probability:

$$\Pr[A \mid B] = \Pr[A \cap B] / \Pr[B]$$

Chain rule:

$$\Pr[A \cap B] = \Pr[A] \cdot \Pr[B \mid A]$$

SUMMARY SO FAR

Real World → Code → Probability Tree

$\parallel$

Mathematical Model

- set of outcomes Ω
- a prob. associated with each outcome.

Events

Conditional probability:

$$\Pr[A \mid B] = \Pr[A \cap B] / \Pr[B]$$

Chain rule:

$$\Pr[A \cap B] = \Pr[A] \cdot \Pr[B \mid A]$$

Independent events:

$$\Pr[A \cap B] = \Pr[A] \cdot \Pr[B]$$

SUMMARY SO FAR

Real World → Code → Probability Tree

$\parallel$

Mathematical Model

- set of outcomes Ω
- a prob. associated with each outcome.

Events

Conditional probability:

$$\Pr[A \mid B] = \Pr[A \cap B] / \Pr[B]$$

Chain rule:

$$\Pr[A \cap B] = \Pr[A] \cdot \Pr[B \mid A]$$

Independent events:

$$\Pr[A \cap B] = \Pr[A] \cdot \Pr[B]$$

Union bound:

$$\Pr[A \cup B] \leq \Pr[A] + \Pr[B]$$

Random Variables

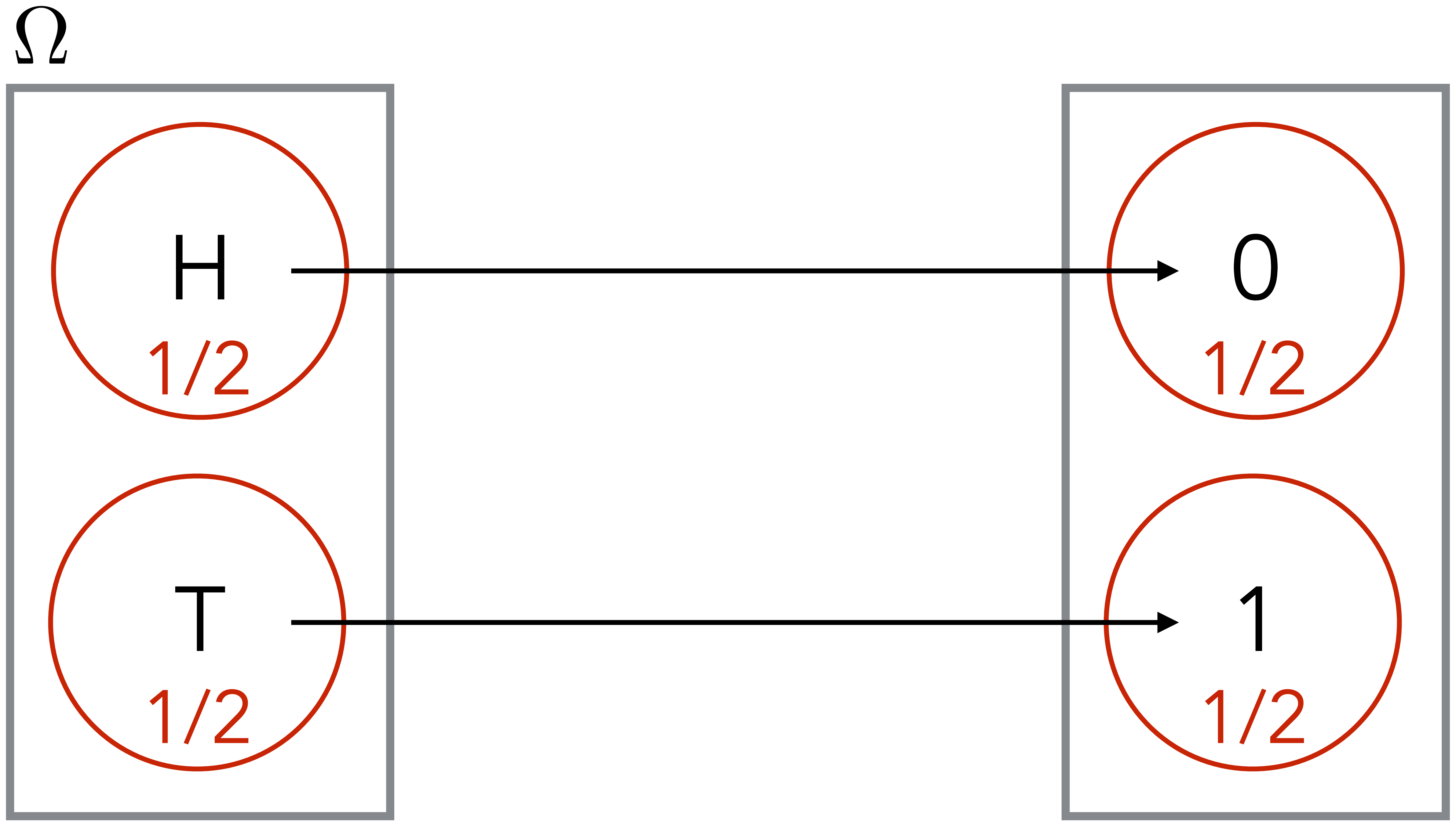
What is a Random Variable?

What is a Random Variable?

Transformation of the sample space to $\mathbb{R}$.

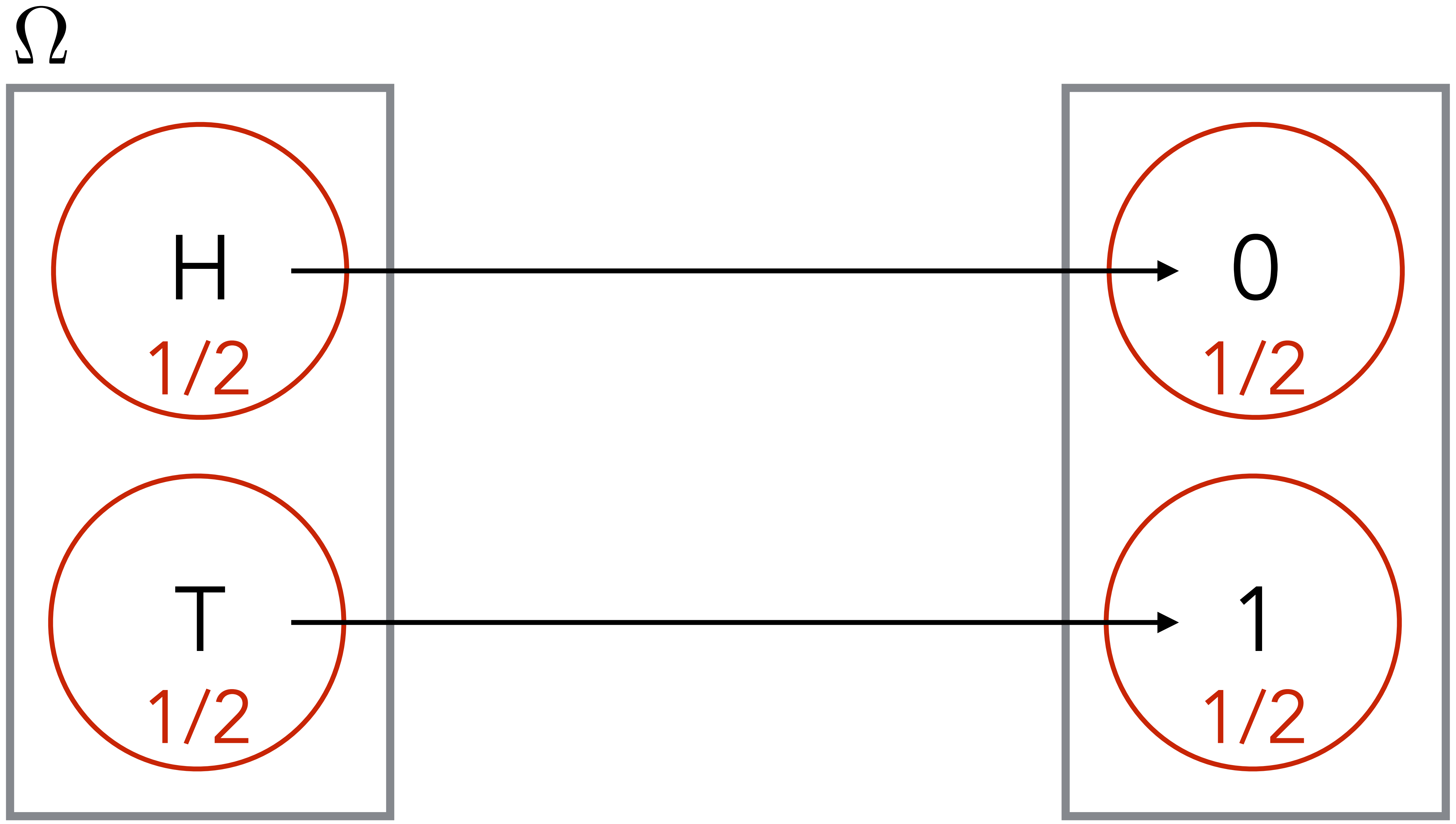
What is a Random Variable?

Transformation of the sample space to $\mathbb{R}$.



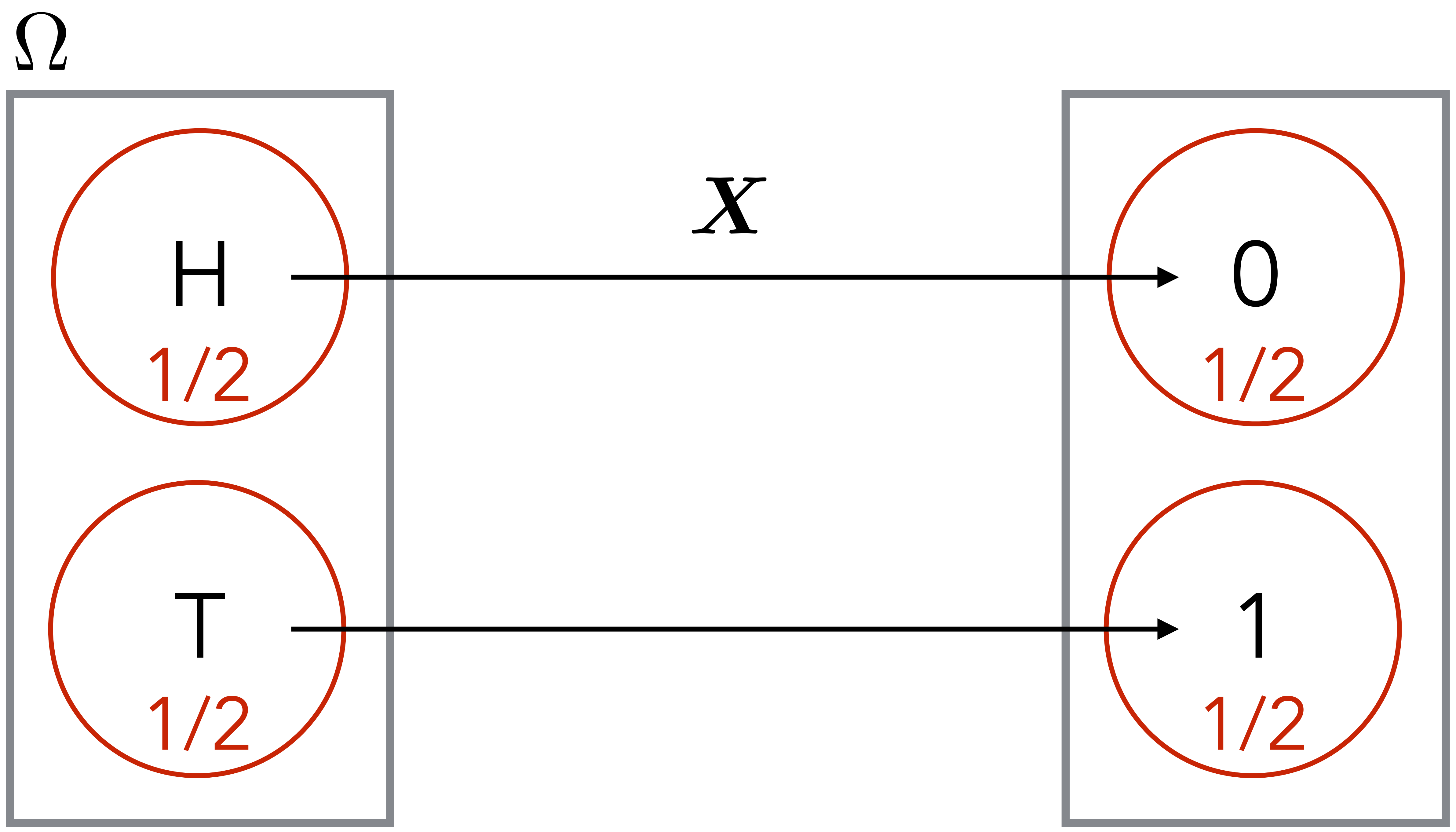
What is a Random Variable?

Transformation of the sample space to $\mathbb{R}$. i.e. a function $\boldsymbol{X} : \Omega \to \mathbb{R}$



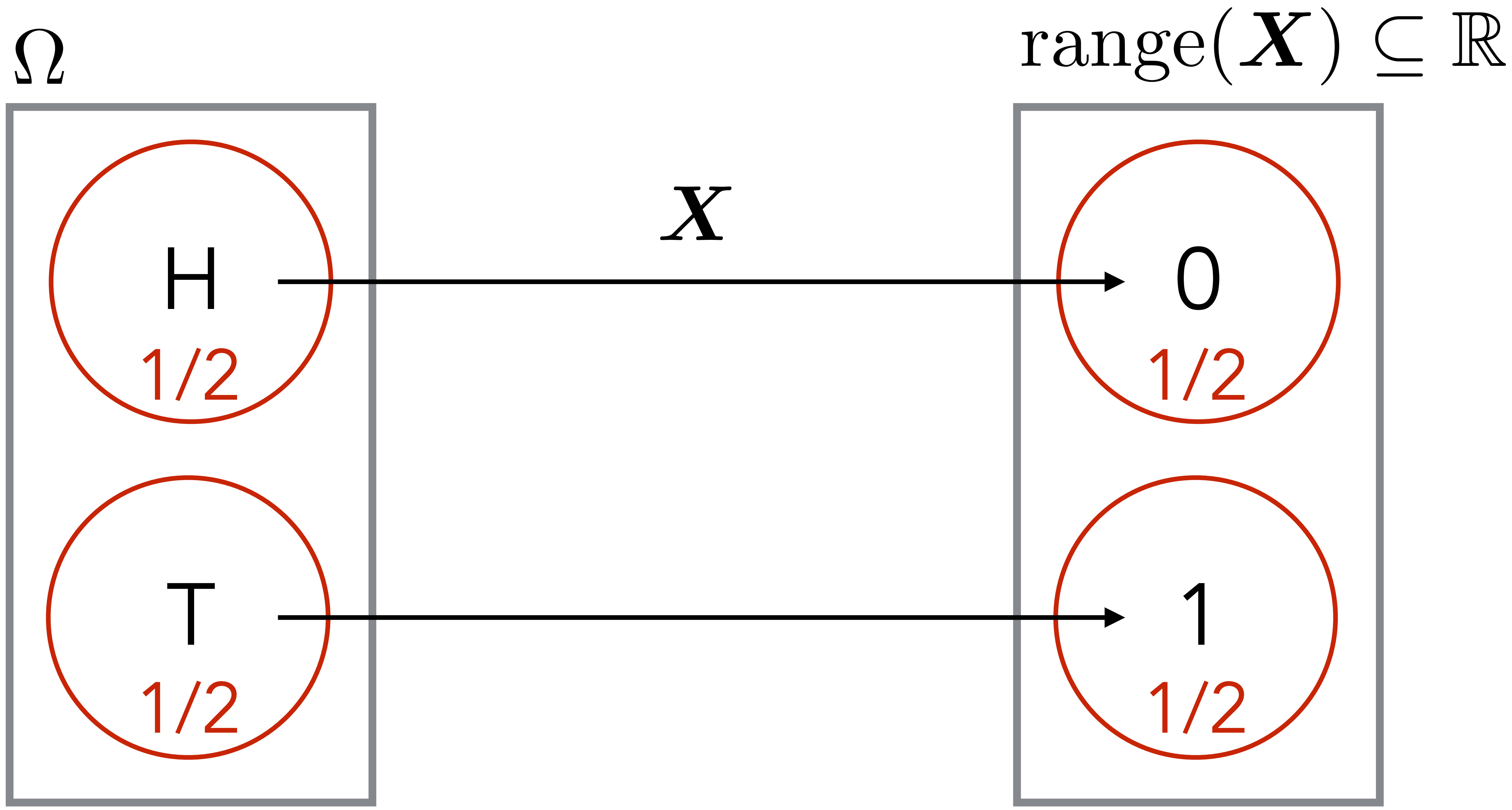
What is a Random Variable?

Transformation of the sample space to $\mathbb{R}$. i.e. a function $\boldsymbol{X} : \Omega \to \mathbb{R}$



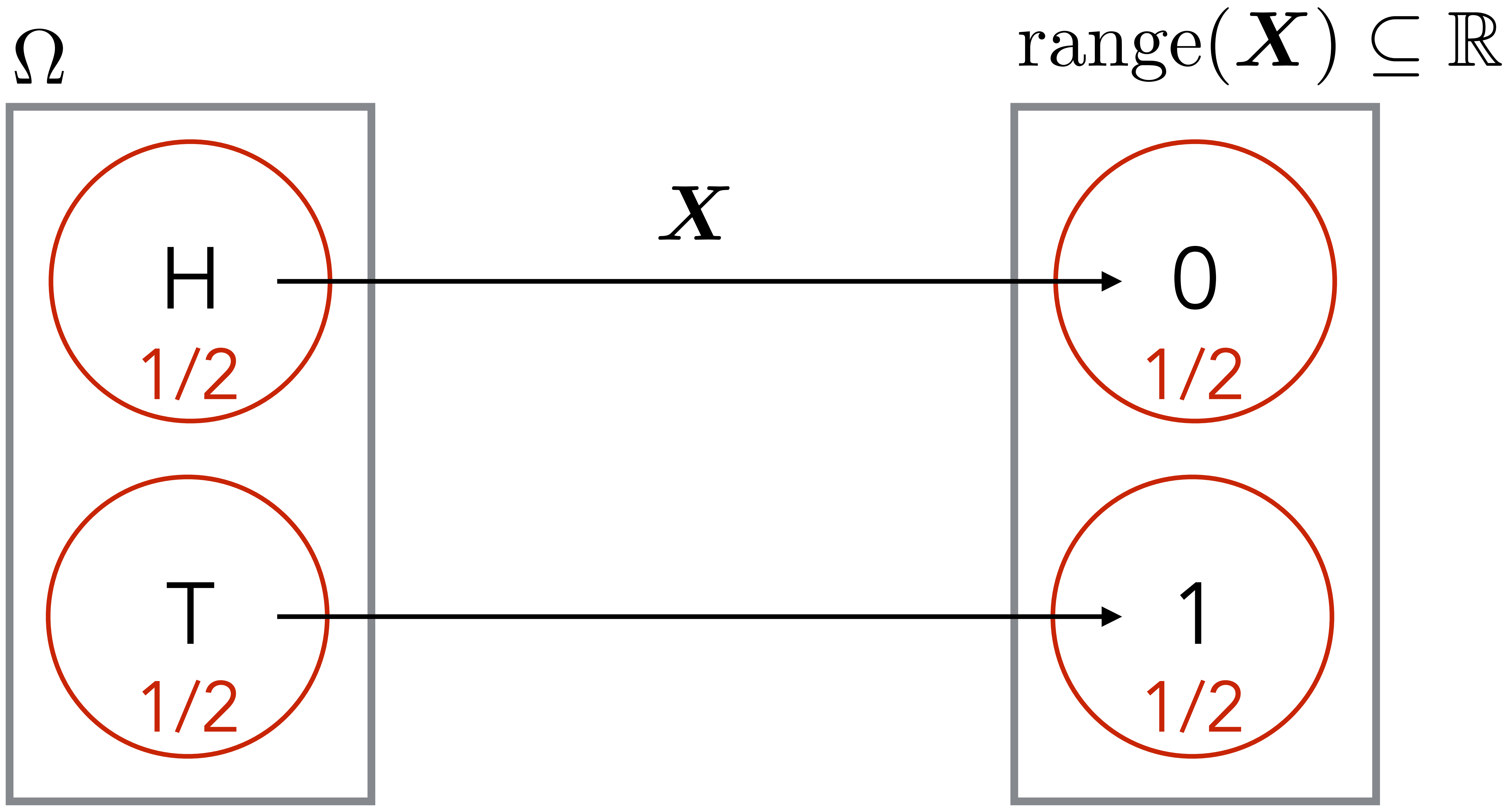
What is a Random Variable?

Transformation of the sample space to $\mathbb{R}$. i.e. a function $\boldsymbol{X} : \Omega \to \mathbb{R}$



What is a Random Variable?

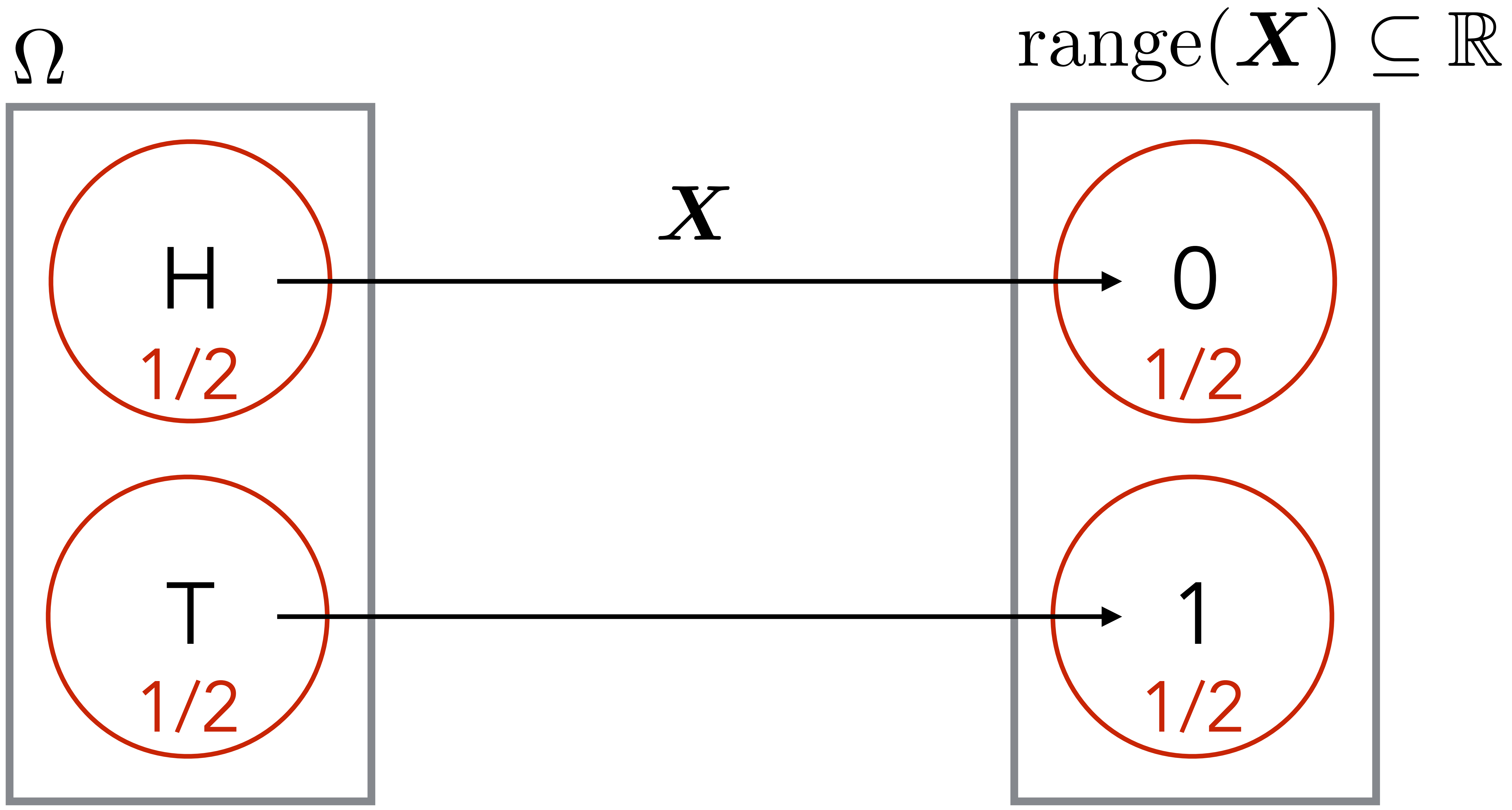
Transformation of the sample space to $\mathbb{R}$.   i.e. a function $\boldsymbol{X} : \Omega \to \mathbb{R}$



<u>typical description:</u> $\boldsymbol{X}$ = number of Tails

What is a Random Variable?

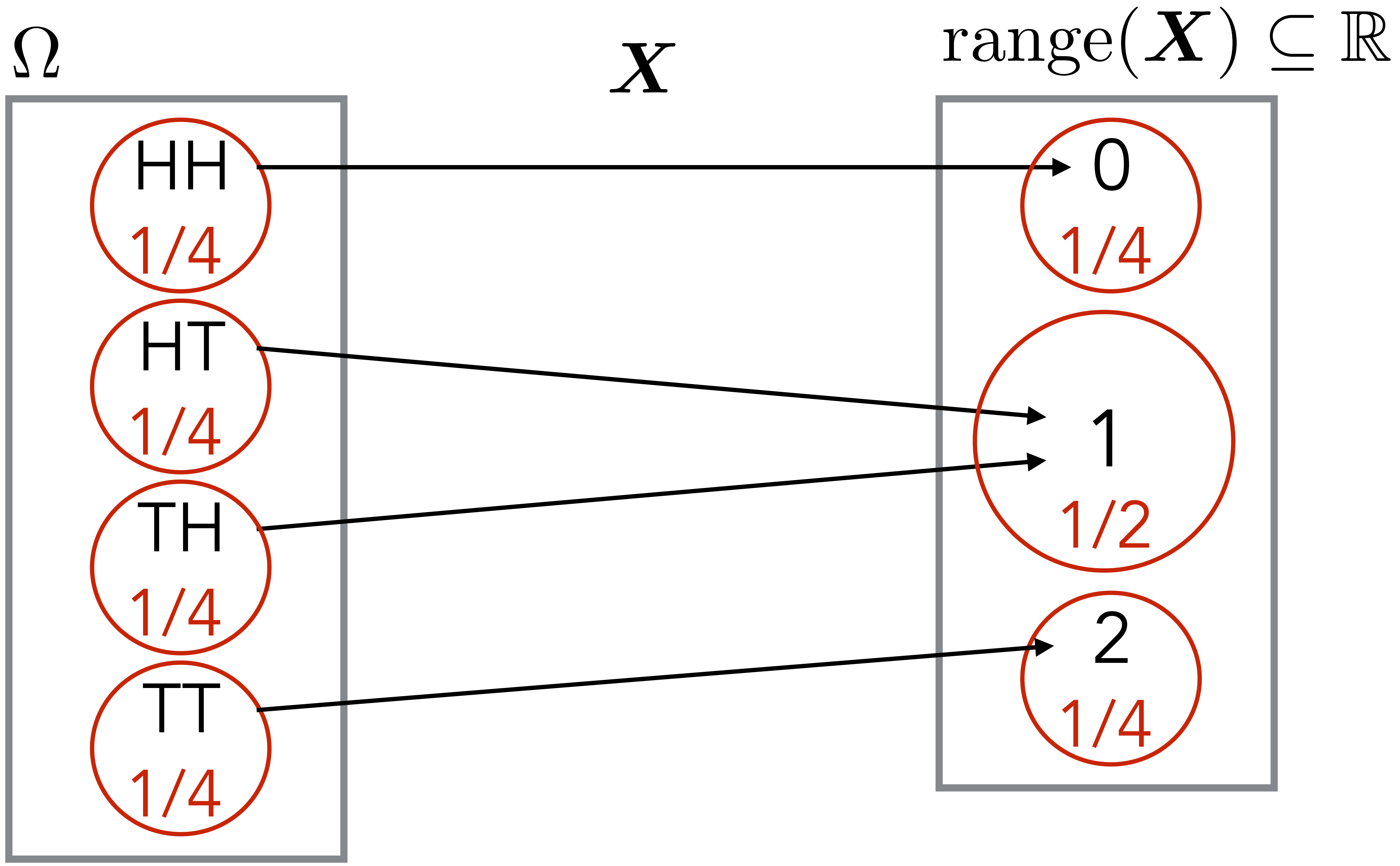
Transformation of the sample space to $\mathbb{R}$. i.e. a function $\boldsymbol{X} : \Omega \to \mathbb{R}$



typical description: $\boldsymbol{X}$ = number of Tails

What is a Random Variable?

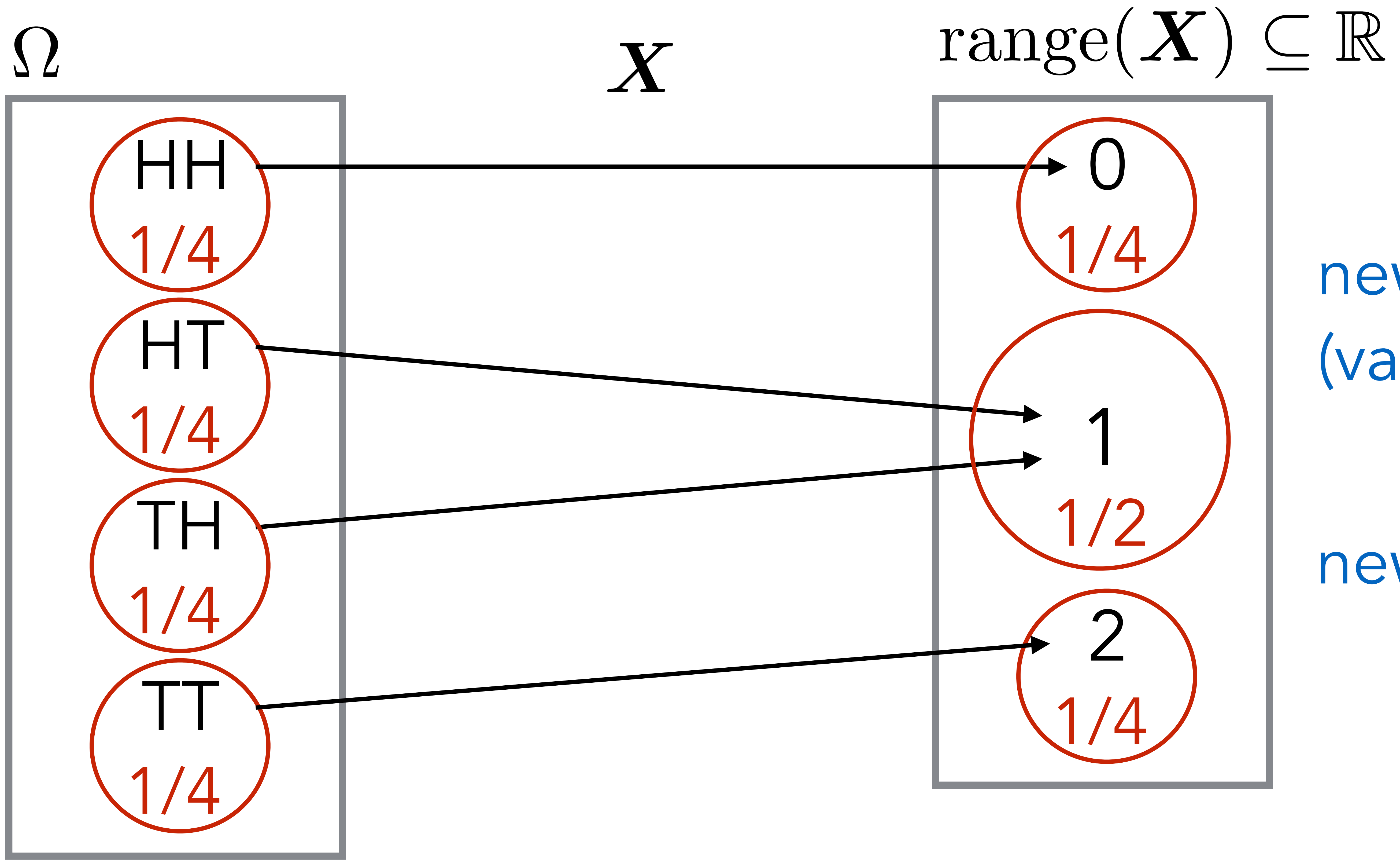
Transformation of the sample space to $\mathbb{R}$. i.e. a function $\boldsymbol{X} : \Omega \to \mathbb{R}$



<u>typical description:</u> $\boldsymbol{X}$ = number of Tails

What is a Random Variable?

Transformation of the sample space to $\mathbb{R}$. i.e. a function $\boldsymbol{X} : \Omega \to \mathbb{R}$



new sample space
(values $\boldsymbol{X}$ can take)

new prob. distribution

<u>typical description:</u> $\boldsymbol{X}$ = number of Tails

What is a Random Variable?

Transformation of the sample space to $\mathbb{R}$.   i.e. a function $\boldsymbol{X} : \Omega \to \mathbb{R}$

Experiment: Roll two dice

What is a Random Variable?

Transformation of the sample space to $\mathbb{R}$. i.e. a function $\boldsymbol{X} : \Omega \to \mathbb{R}$

Experiment: Roll two dice

$\Omega =$ {(1,1), (1,2), (1,3), (1,4), (1,5), (1,6),
(2,1), (2,2), (2,3), (2,4), (2,5), (2,6),
(3,1), (3,2), (3,3), (3,4), (3,5), (3,6),
(4,1), (4,2), (4,3), (4,4), (4,5), (4,6),
(5,1), (5,2), (5,3), (5,4), (5,5), (5,6),
(6,1), (6,2), (6,3), (6,4), (6,5), (6,6)}

What is a Random Variable?

Transformation of the sample space to $\mathbb{R}$. i.e. a function $\boldsymbol{X} : \Omega \to \mathbb{R}$

Experiment: Roll two dice

$\Omega =$ {(1,1), (1,2), (1,3), (1,4), (1,5), (1,6),
(2,1), (2,2), (2,3), (2,4), (2,5), (2,6),
(3,1), (3,2), (3,3), (3,4), (3,5), (3,6),
(4,1), (4,2), (4,3), (4,4), (4,5), (4,6),
(5,1), (5,2), (5,3), (5,4), (5,5), (5,6),
(6,1), (6,2), (6,3), (6,4), (6,5), (6,6)}

Distribution:

What is a Random Variable?

Transformation of the sample space to $\mathbb{R}$. i.e. a function $\boldsymbol{X} : \Omega \to \mathbb{R}$

Experiment: Roll two dice

$\Omega =$ {(1,1), (1,2), (1,3), (1,4), (1,5), (1,6),
(2,1), (2,2), (2,3), (2,4), (2,5), (2,6),
(3,1), (3,2), (3,3), (3,4), (3,5), (3,6),
(4,1), (4,2), (4,3), (4,4), (4,5), (4,6),
(5,1), (5,2), (5,3), (5,4), (5,5), (5,6),
(6,1), (6,2), (6,3), (6,4), (6,5), (6,6)}

Distribution:

For each $\ell \in \Omega :\quad \Pr[\ell] = 1/36$

('uniform distribution')

What is a Random Variable?

Transformation of the sample space to $\mathbb{R}$. i.e. a function $\boldsymbol{X} : \Omega \to \mathbb{R}$

Experiment: Roll two dice

$$\Omega = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6),\\ (2,1), (2,2), (2,3), (2,4), (2,5), (2,6),\\ (3,1), (3,2), (3,3), (3,4), (3,5), (3,6),\\ (4,1), (4,2), (4,3), (4,4), (4,5), (4,6),\\ (5,1), (5,2), (5,3), (5,4), (5,5), (5,6),\\ (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}$$

Distribution:

For each $\ell \in \Omega :\quad \Pr[\ell] = 1/36$

('uniform distribution')

What is a Random Variable?

Transformation of the sample space to $\mathbb{R}$. i.e. a function $\boldsymbol{X} : \Omega \to \mathbb{R}$

Experiment: Roll two dice

$\Omega =$ {(1,1), (1,2), (1,3), (1,4), (1,5), (1,6),
(2,1), (2,2), (2,3), (2,4), (2,5), (2,6),
(3,1), (3,2), (3,3), (3,4), (3,5), (3,6),
(4,1), (4,2), (4,3), (4,4), (4,5), (4,6),
(5,1), (5,2), (5,3), (5,4), (5,5), (5,6),
(6,1), (6,2), (6,3), (6,4), (6,5), (6,6)}

Distribution:

For each $\ell \in \Omega$: $\Pr[\ell] = 1/36$

('uniform distribution')

$\boldsymbol{X}$ = sum of the two dice

What is a Random Variable?

Transformation of the sample space to $\mathbb{R}$. i.e. a function $\boldsymbol{X} : \Omega \to \mathbb{R}$

Experiment: Roll two dice

$$\Omega = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), \\ (2,1), (2,2), (2,3), (2,4), (2,5), (2,6), \\ (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), \\ (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), \\ (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), \\ (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}$$

Distribution:

For each $\ell \in \Omega$: $\Pr[\ell] = 1/36$

('uniform distribution')

$\boldsymbol{X}$ = sum of the two dice

$\text{range}(\boldsymbol{X})$ = {2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}

What is a Random Variable?

Transformation of the sample space to $\mathbb{R}$. i.e. a function $\boldsymbol{X} : \Omega \to \mathbb{R}$

Experiment: Roll two dice

$\Omega = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6),$
$(2,1), (2,2), (2,3), (2,4), (2,5), (2,6),$
$(3,1), (3,2), (3,3), (3,4), (3,5), (3,6),$
$(4,1), (4,2), (4,3), (4,4), (4,5), (4,6),$
$(5,1), (5,2), (5,3), (5,4), (5,5), (5,6),$
$(6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}$

Distribution:

For each $\ell \in \Omega$: $\Pr[\ell] = 1/36$

('uniform distribution')

$\boldsymbol{X}$ = sum of the two dice

$\text{range}(\boldsymbol{X})$ = {2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}

Distribution:

What is a Random Variable?

Transformation of the sample space to $\mathbb{R}$. i.e. a function $\boldsymbol{X} : \Omega \to \mathbb{R}$

Experiment: Roll two dice

$$\Omega = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,2), (2,3), (2,4), (2,5), (2,6), (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}$$

Distribution:

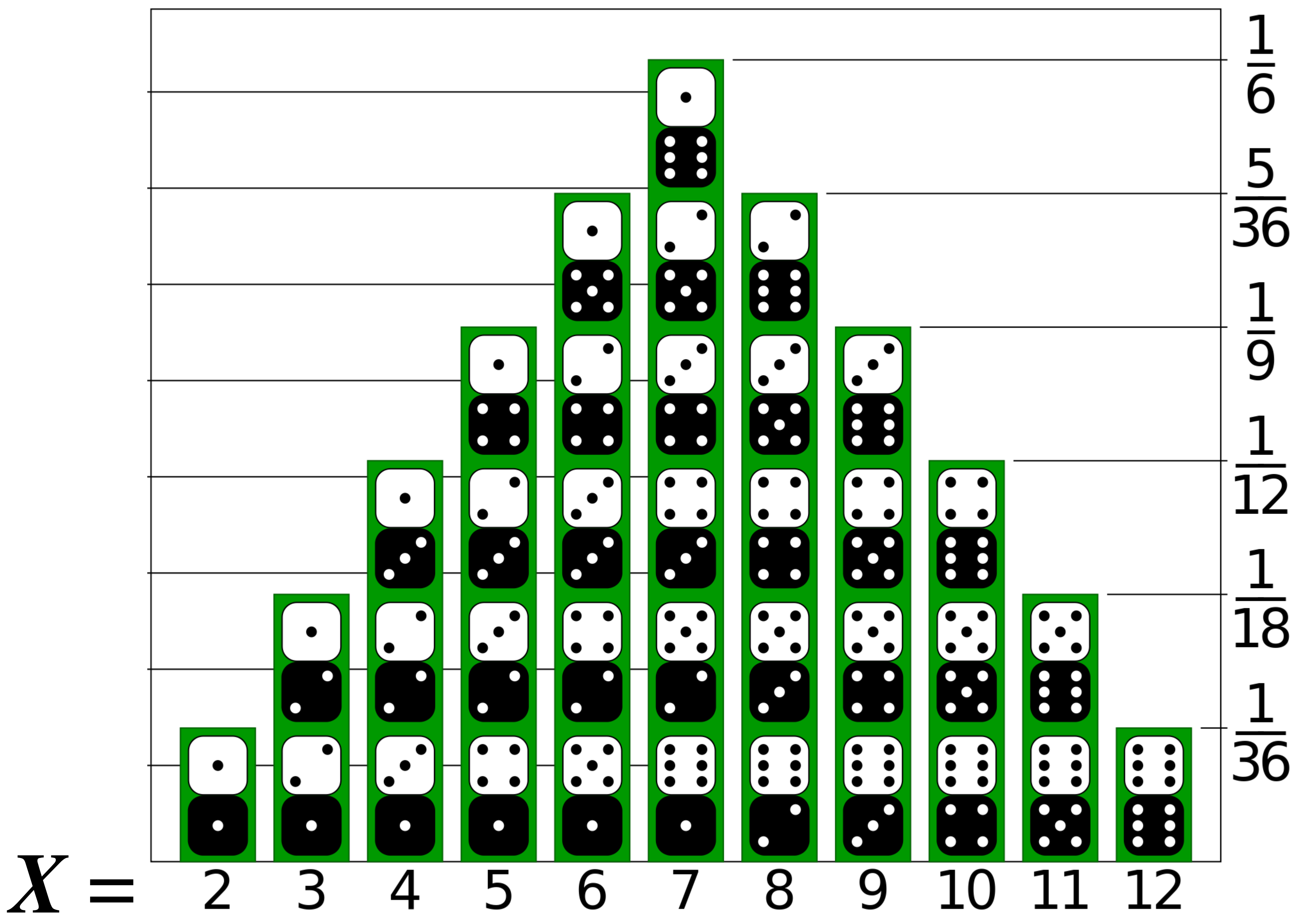
For each $\ell \in \Omega$: $\Pr[\ell] = 1/36$

('uniform distribution')

$\boldsymbol{X}$ = sum of the two dice

$\mathrm{range}(\boldsymbol{X}) = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

Distribution:



Why?

1.

2.

Why?

1. Often interested in ***numerical*** outcomes

   (e.g. # Tails we see if we toss n coins)

2.

Why?

1. Often interested in ***numerical*** outcomes

   (e.g. # Tails we see if we toss *n* coins)

   but initially outcomes are best expressed ***non-numerically***.

   (e.g. an outcome is a sequence of *n* coin tosses)

2.

Why?

1. Often interested in ***numerical*** outcomes

   (e.g. # Tails we see if we toss *n* coins)

   but initially outcomes are best expressed ***non-numerically***.

   (e.g. an outcome is a sequence of *n* coin tosses)

2. We like talking about mean values (averages), variance, etc.

What is a Random Variable?

<u>2nd (CS) Definition:</u>

What is a Random Variable?

2nd (CS) Definition:

A **random variable** is a variable in some randomized code

(the variable's value at the end of the execution)

What is a Random Variable?

2nd (CS) Definition:

A **random variable** is a variable in some randomized code

(the variable's value at the end of the execution)

of type 'real number'.

What is a Random Variable?

2nd (CS) Definition:

A **random variable** is a variable in some randomized code

(the variable's value at the end of the execution)

of type 'real number'.

Example:

```
S = RandInt(6) + RandInt(6)
if  S == 12:   I = 1
else:          I = 0
```

What is a Random Variable?

2nd (CS) Definition:

A **random variable** is a variable in some randomized code
(the variable's value at the end of the execution)
of type 'real number'.

Example:

```
S = RandInt(6) + RandInt(6)
if  S == 12:   I = 1
else:          I = 0
```

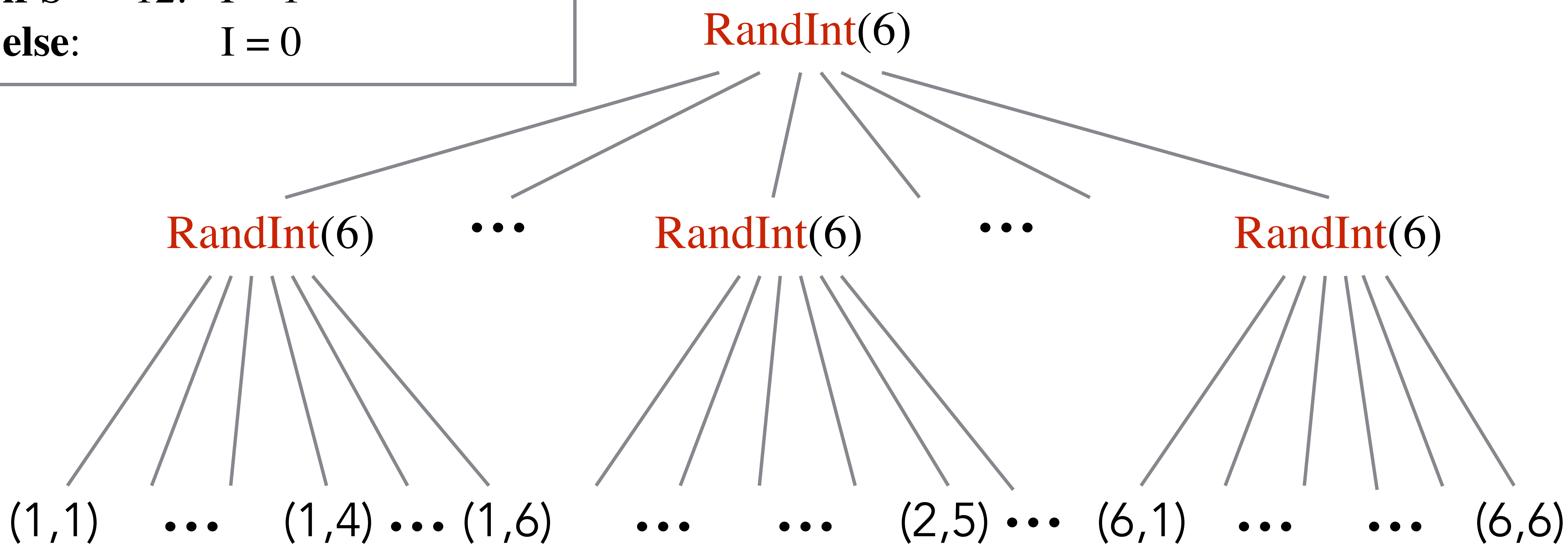
Random variables: ***S*** and ***I***

What is a Random Variable?

```
S = RandInt(6) + RandInt(6)
if S == 12:   I = 1
else:         I = 0
```

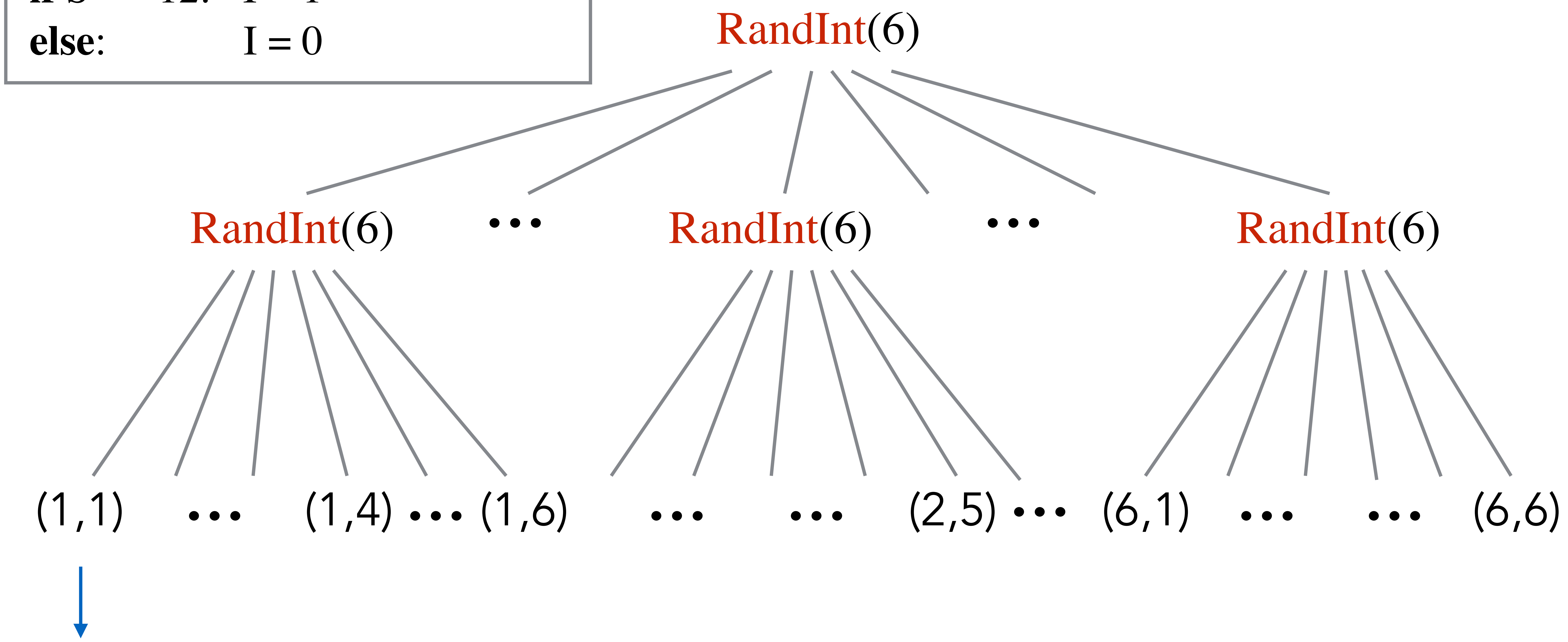
What is a Random Variable?

S = RandInt(6) + RandInt(6)
if S == 12:   I = 1
else: I = 0



What is a Random Variable?

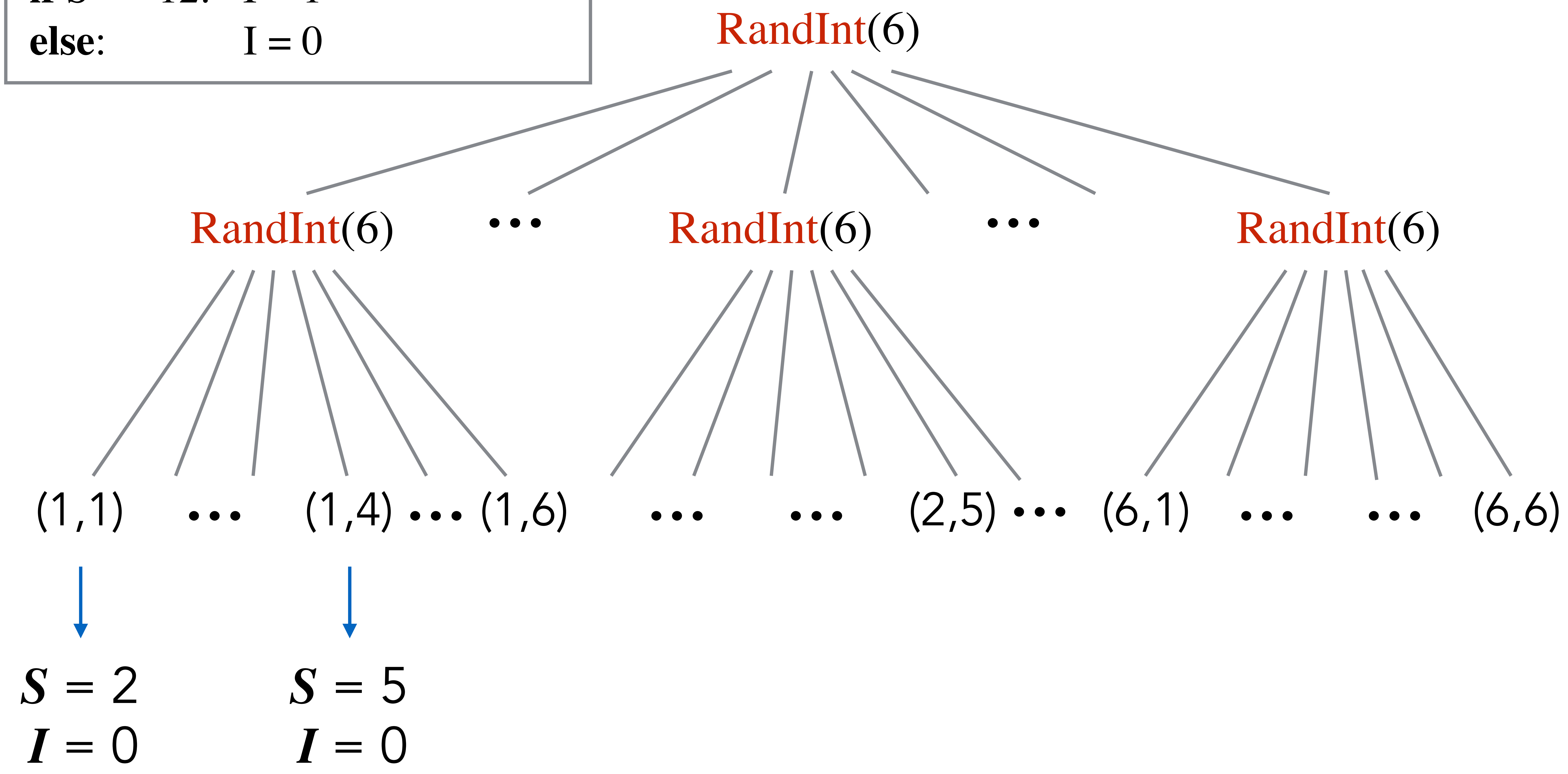
S = RandInt(6) + RandInt(6)
if S == 12: I = 1
else: I = 0



$\boldsymbol{S} = 2$
$\boldsymbol{I} = 0$

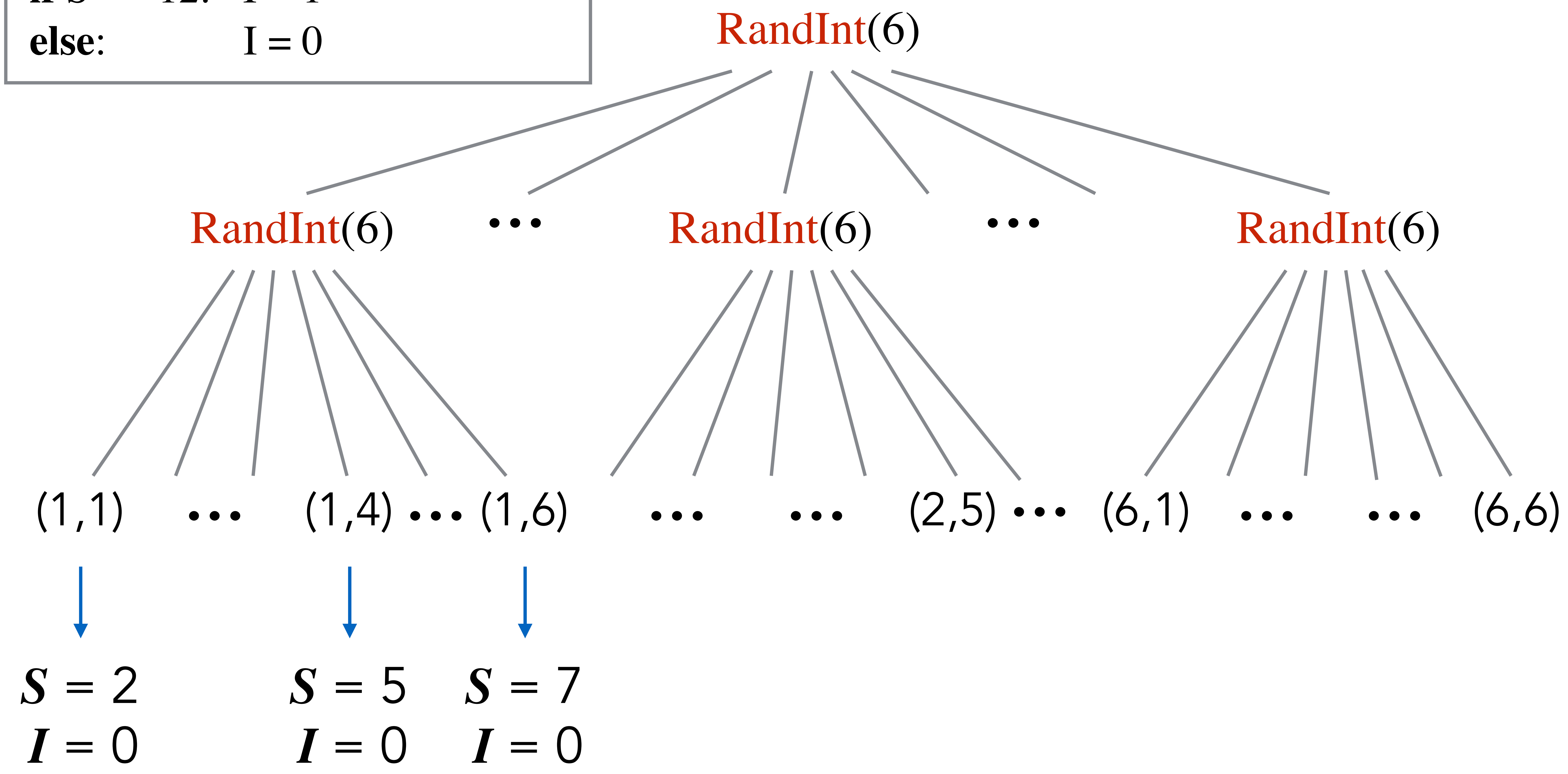
What is a Random Variable?

```
S = RandInt(6) + RandInt(6)
if S == 12:   I = 1
else:         I = 0
```



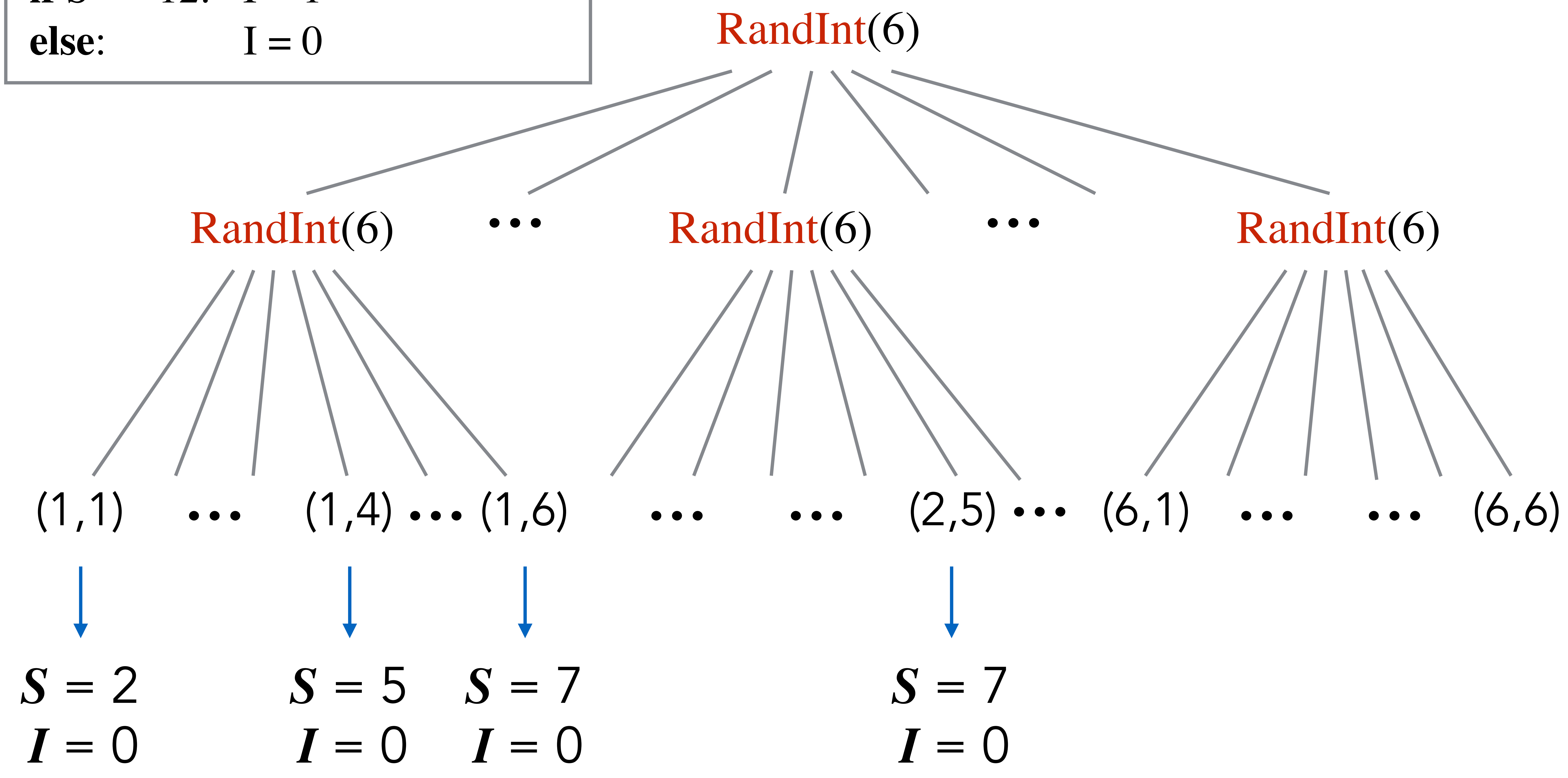
What is a Random Variable?

```
S = RandInt(6) + RandInt(6)
if S == 12:   I = 1
else:         I = 0
```



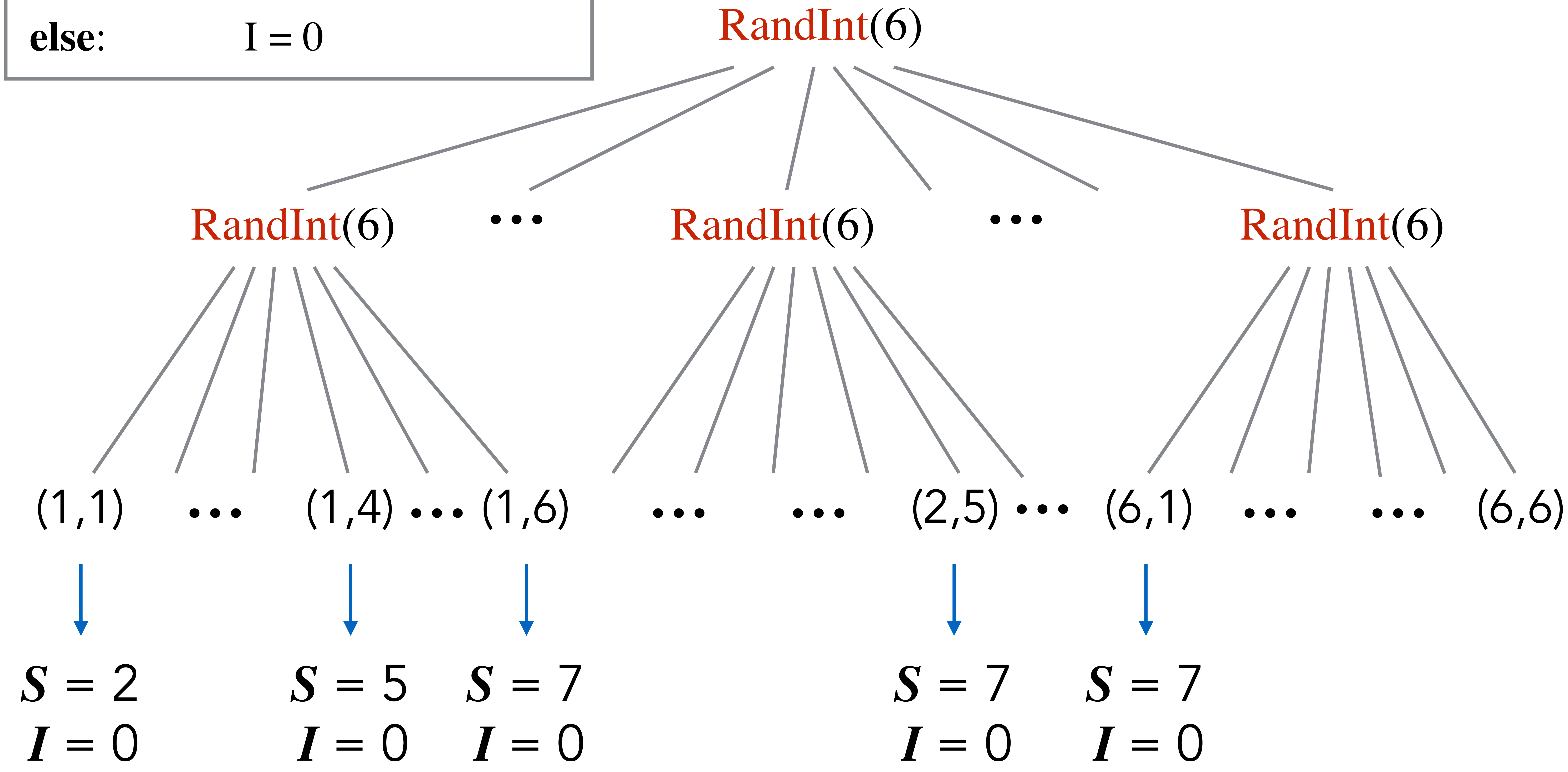
What is a Random Variable?

```
S = RandInt(6) + RandInt(6)
if S == 12:   I = 1
else:         I = 0
```



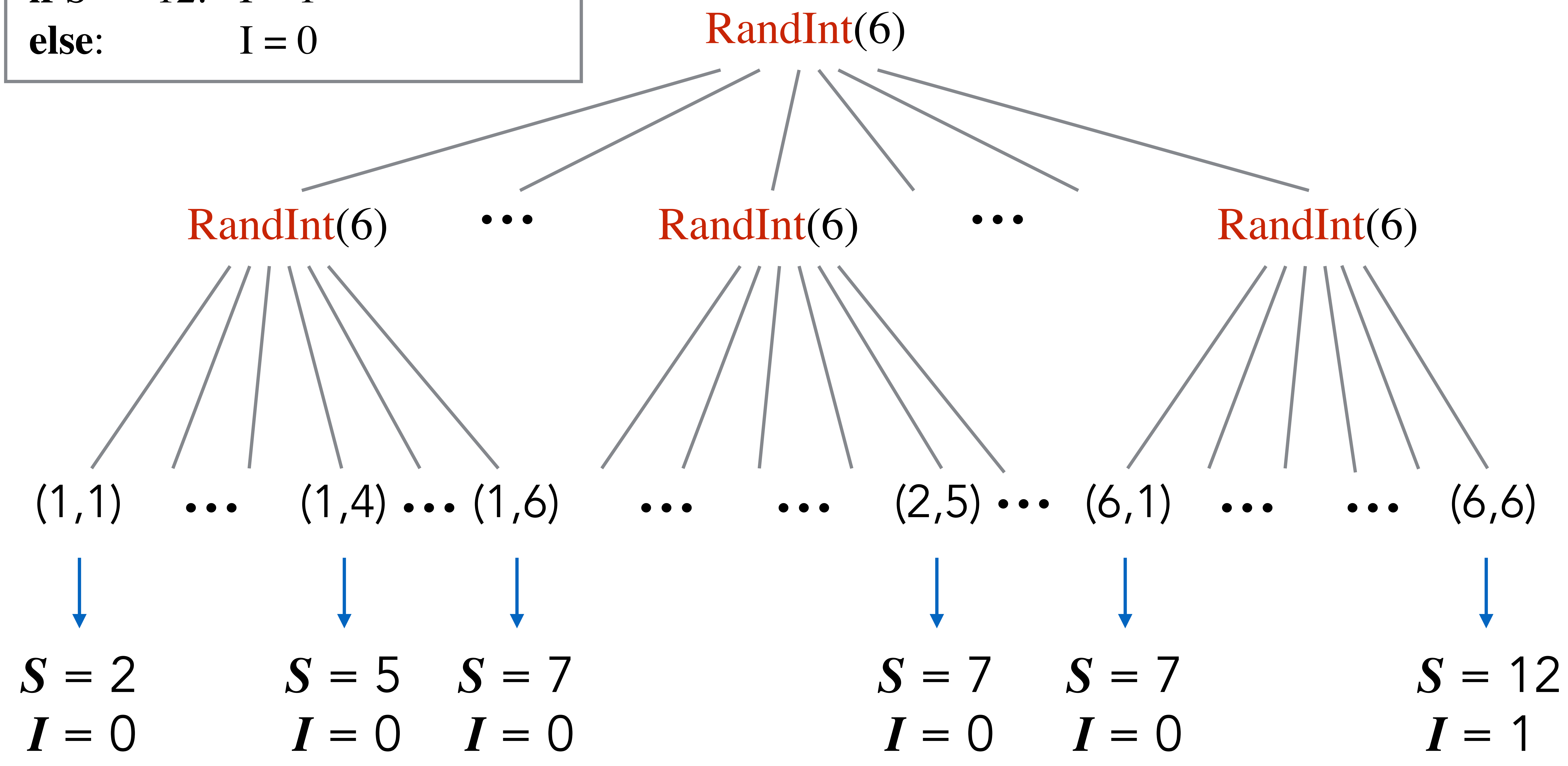
What is a Random Variable?

S = RandInt(6) + RandInt(6)
if S == 12:   I = 1
else: I = 0



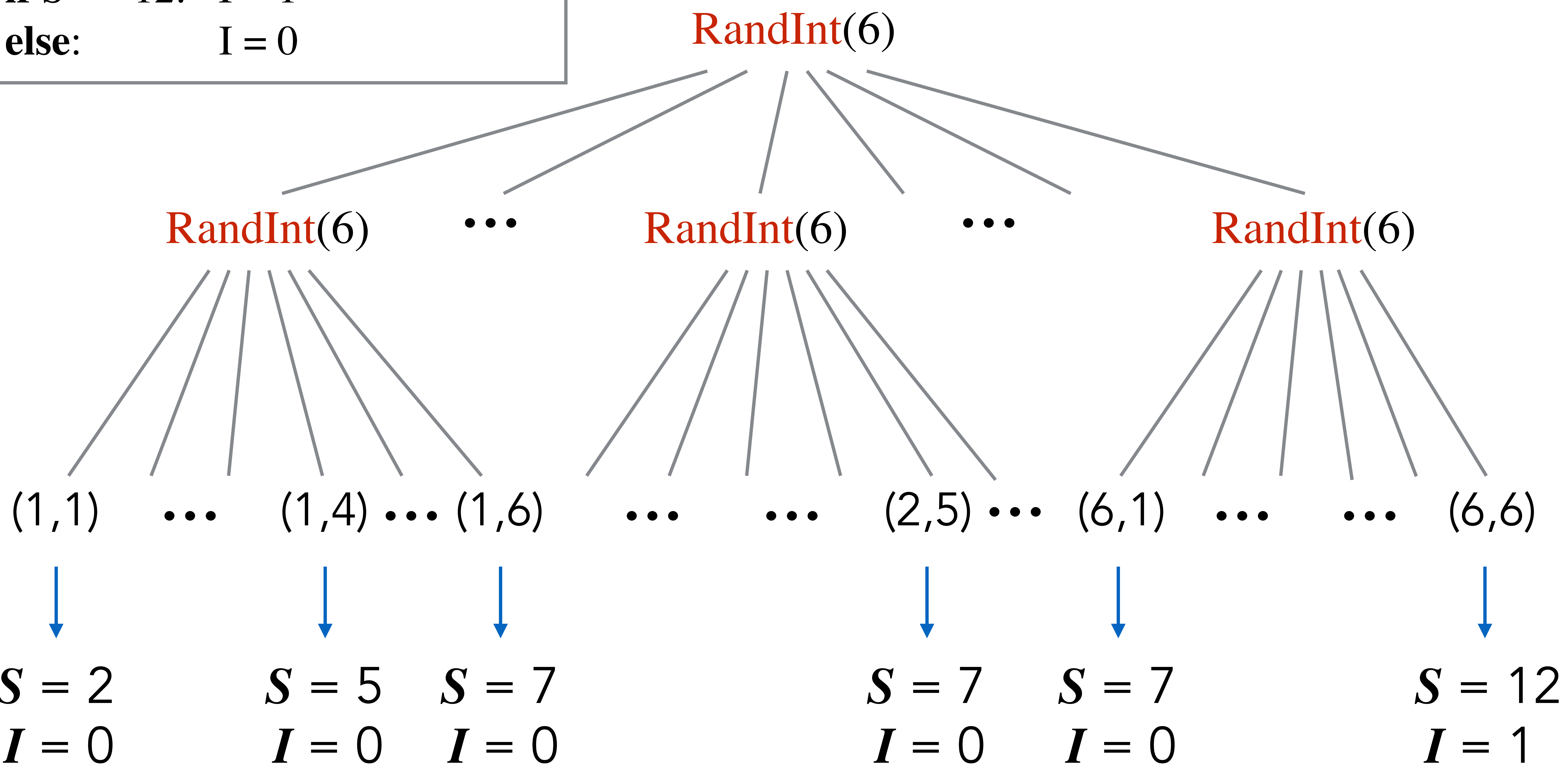
What is a Random Variable?

S = RandInt(6) + RandInt(6)
if S == 12:   I = 1
else: I = 0



What is a Random Variable?

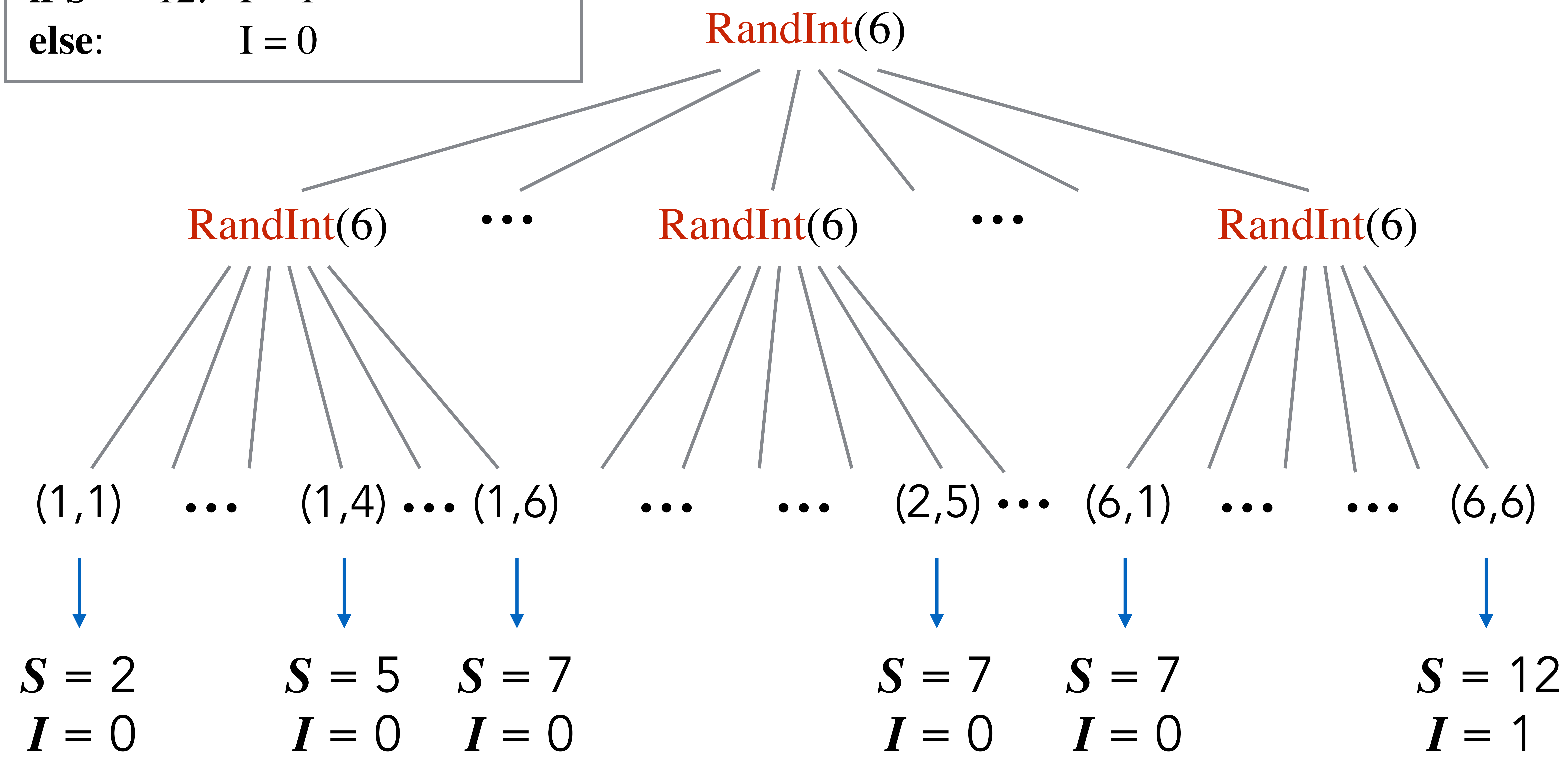
S = RandInt(6) + RandInt(6)
if S == 12:   I = 1
else: I = 0



$\boldsymbol{S} = 7$ is an event.

What is a Random Variable?

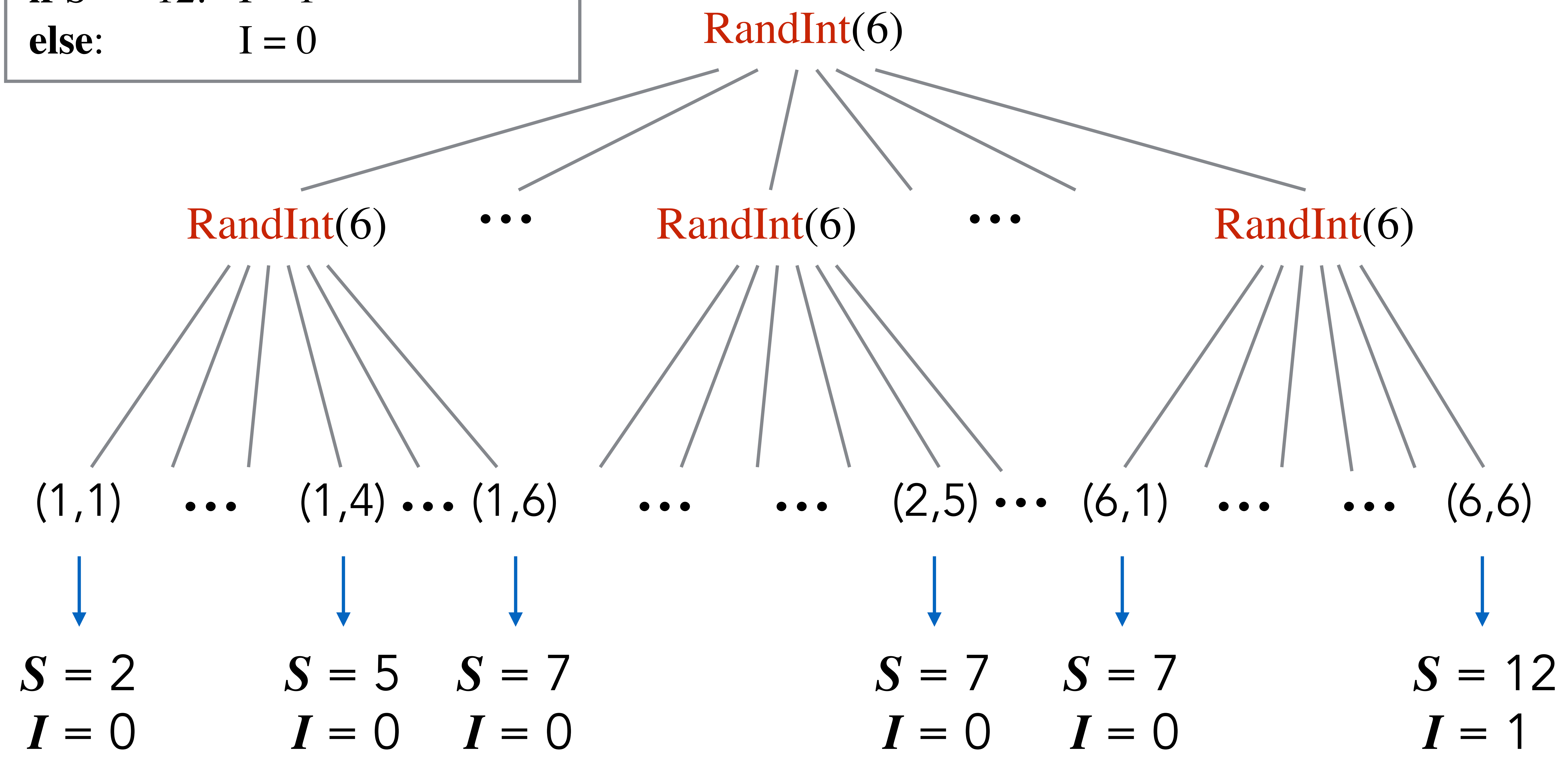
S = RandInt(6) + RandInt(6)
if S == 12:   I = 1
else: I = 0



$S = 7$ is an event. We can ask, what is $\Pr[S = 7]$?

What is a Random Variable?

S = RandInt(6) + RandInt(6)
if S == 12:   I = 1
else: I = 0



$S \geq 7$ is an event.   We can ask, what is $\Pr[S \geq 7]$?

Expectation of a Random Variable

Expectation of a Random Variable

Expected Value = Mean = (Weighted) Average

Expectation of a Random Variable

Expected Value = Mean = (Weighted) Average

Example:

Weight	Value
30% Final	85
20% Midterm	75
50% Homework	82

Expectation of a Random Variable

Expected Value = Mean = (Weighted) Average

Example:

Weight	Value
30% Final	85
20% Midterm	75
50% Homework	82

Weighted Average =

Expectation of a Random Variable

Expected Value = Mean = (Weighted) Average

Example:

Weight	Value
30% Final	85
20% Midterm	75
50% Homework	82

Weighted Average = $0.3 \cdot 85 + 0.2 \cdot 75 + 0.5 \cdot 82$

Expectation of a Random Variable

Expected Value = Mean = (Weighted) Average

Example:

Weight	Value
30% Final	85
20% Midterm	75
50% Homework	82

Weighted Average = $0.3 \cdot 85 + 0.2 \cdot 75 + 0.5 \cdot 82 = 81.5$

Expectation of a Random Variable

Expected Value = Mean = (Weighted) Average

Example:

Weight	Value
30% Final	85
20% Midterm	75
50% Homework	82

Weighted Average = $0.3 \cdot 85 + 0.2 \cdot 75 + 0.5 \cdot 82 = 81.5$

Weighted Average = $\displaystyle\sum_{\text{elements } e} \text{value}(e) \cdot \text{weight}(e)$

Expectation of a Random Variable

Expected value of a random variable X:

Expectation of a Random Variable

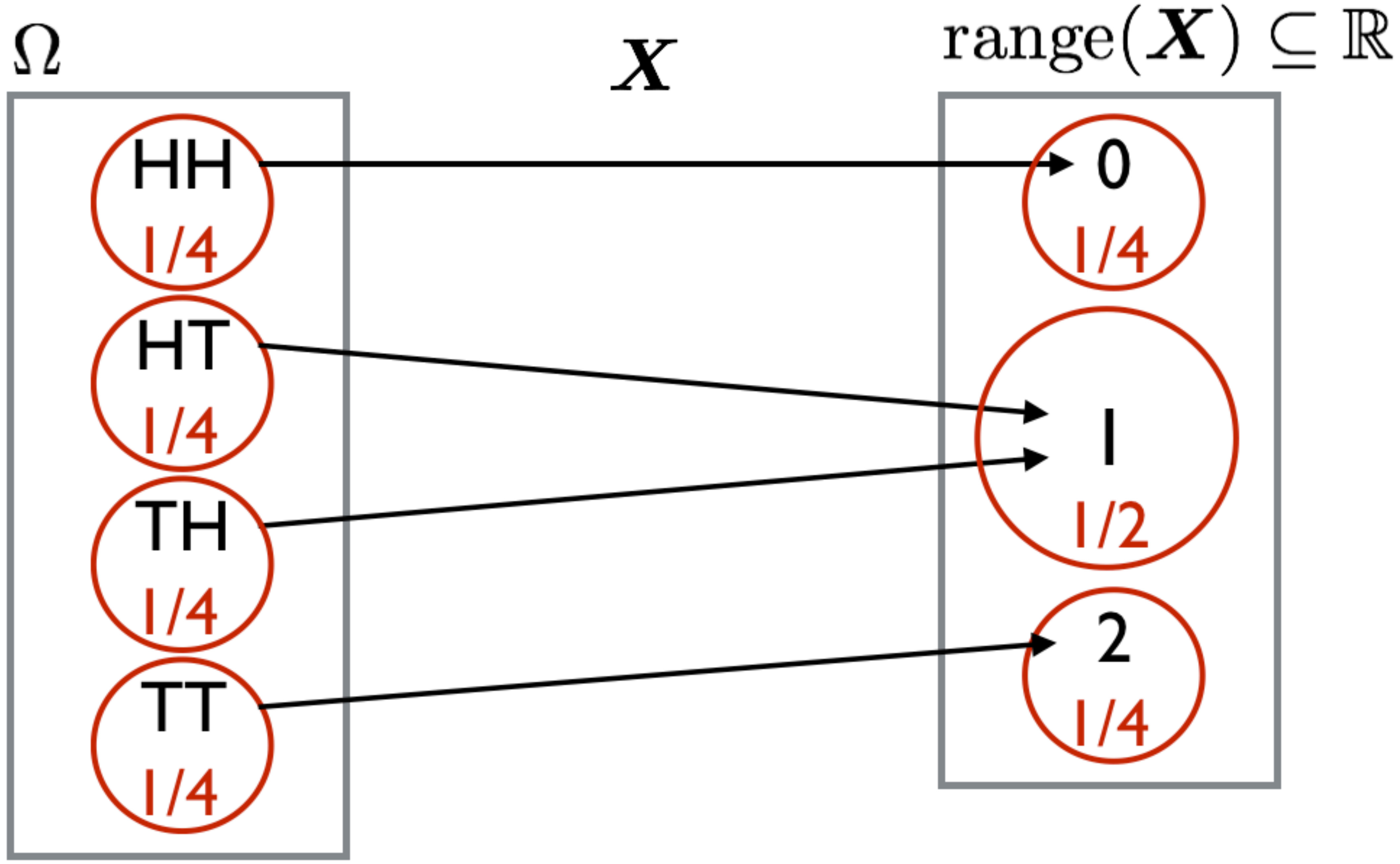
Expected value of a random variable $\boldsymbol{X}$:

$$\mathbf{E}[\boldsymbol{X}] \stackrel{\text{def}}{=} \sum_{x \in \mathrm{range}(\boldsymbol{X})} x \cdot \Pr[\boldsymbol{X} = x]$$

Expectation of a Random Variable

Expected value of a random variable $\boldsymbol{X}$:

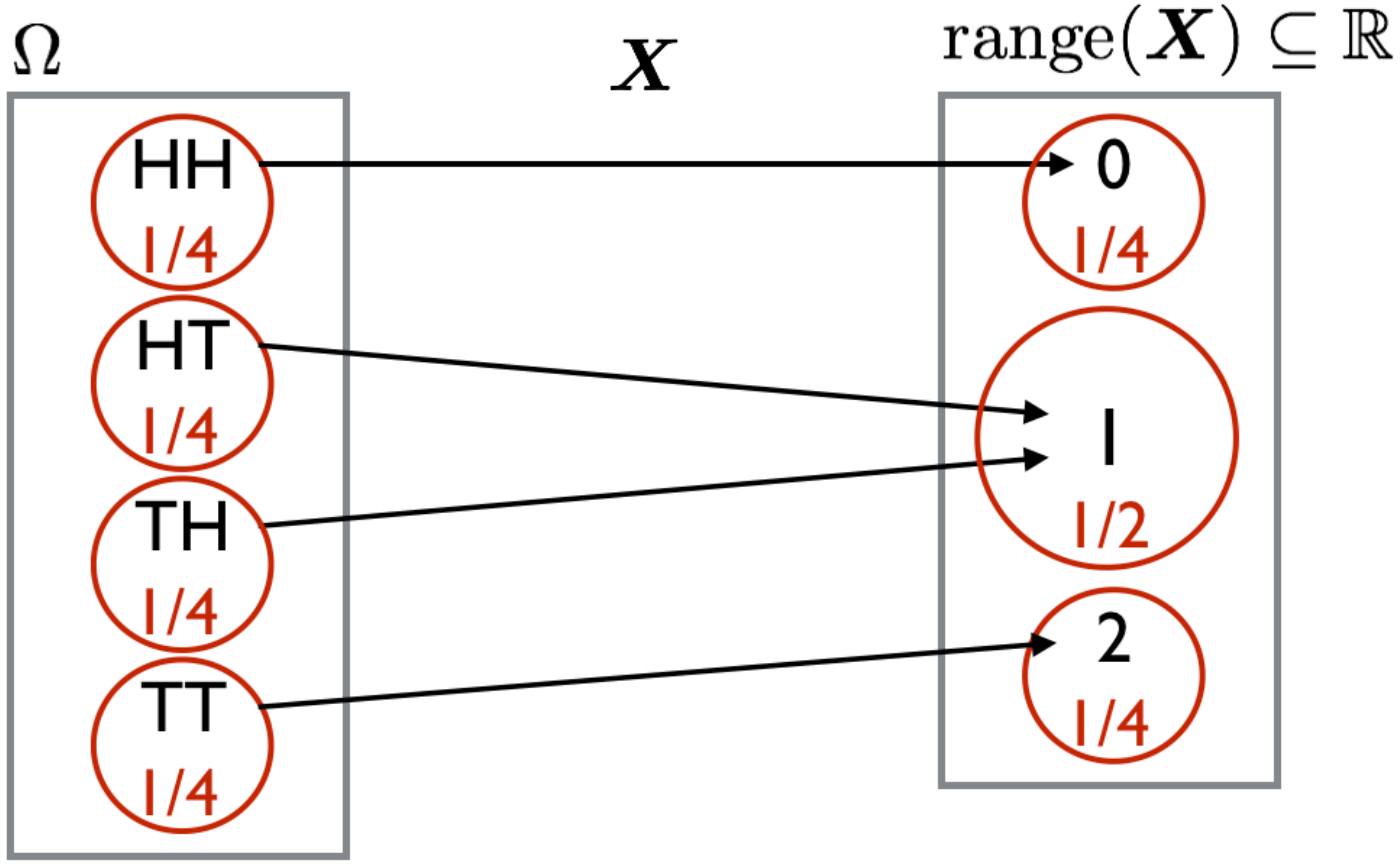
$$\mathbf{E}[\boldsymbol{X}] \stackrel{\text{def}}{=} \sum_{x \in \text{range}(\boldsymbol{X})} x \cdot \Pr[\boldsymbol{X} = x]$$



Expectation of a Random Variable

Expected value of a random variable $\boldsymbol{X}$:

$$\mathbf{E}[\boldsymbol{X}] \stackrel{\text{def}}{=} \sum_{x \in \mathrm{range}(\boldsymbol{X})} x \cdot \Pr[\boldsymbol{X} = x]$$



0 x 1/4 +

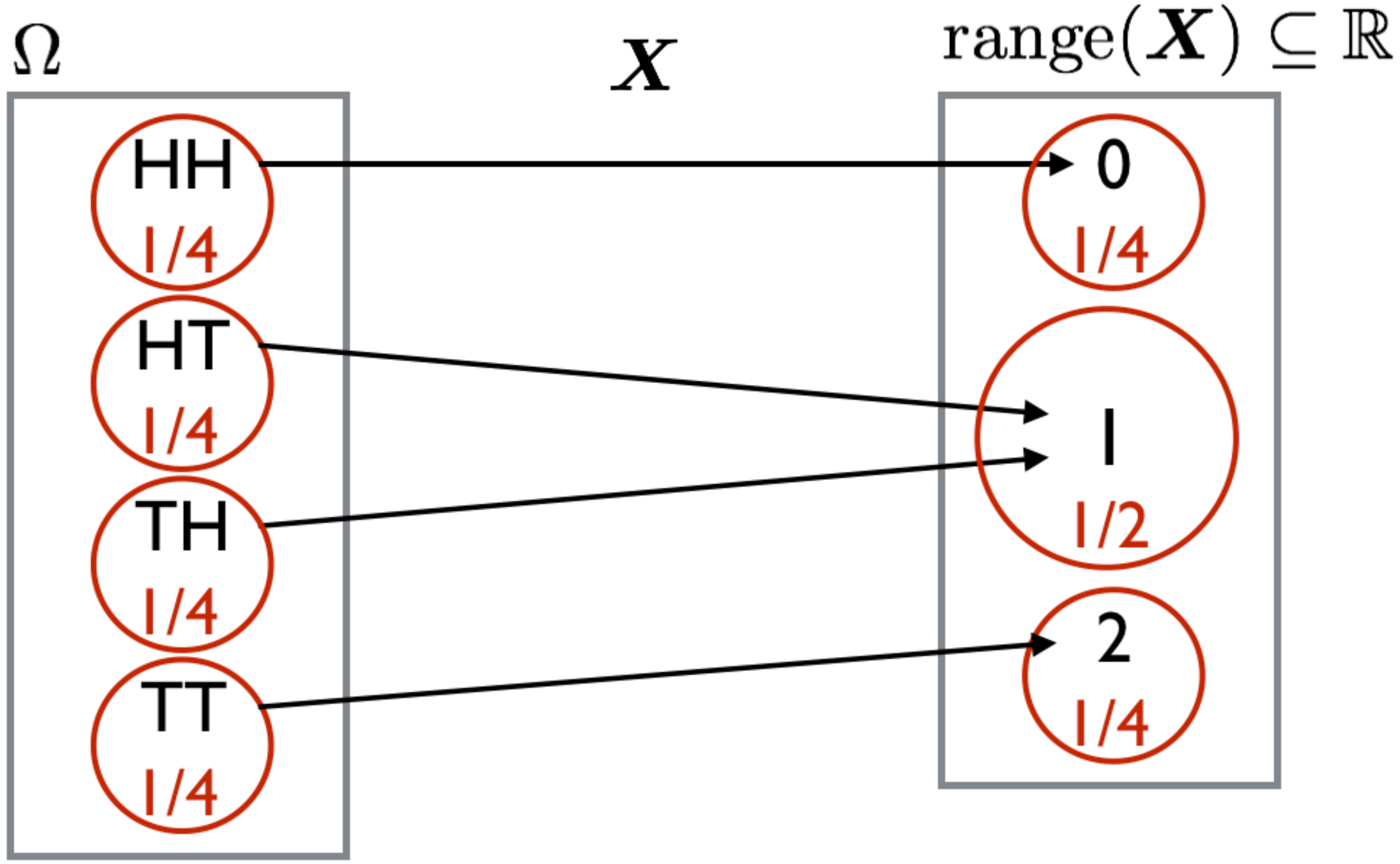
1 x 1/2 +

2 x 1/4

Expectation of a Random Variable

Expected value of a random variable $\boldsymbol{X}$:

$$\mathbf{E}[\boldsymbol{X}] \stackrel{\text{def}}{=} \sum_{x \in \text{range}(\boldsymbol{X})} x \cdot \Pr[\boldsymbol{X} = x]$$



0 x 1/4 +

1 x 1/2 +

2 x 1/4 = 1

Expectation of a Random Variable

Let X be the outcome of the roll of a 6-sided die.

X = RandInt(6)

Expectation of a Random Variable

Let $\boldsymbol{X}$ be the outcome of the roll of a 6-sided die.

$\boldsymbol{X}$ = RandInt(6)

$\mathbf{E}[\boldsymbol{X}]$

Expectation of a Random Variable

Let $\boldsymbol{X}$ be the outcome of the roll of a 6-sided die.

$\boldsymbol{X}$ = RandInt(6)

$$\mathbf{E}[\boldsymbol{X}] = 1 \cdot \Pr[\boldsymbol{X} = 1] + 2 \cdot \Pr[\boldsymbol{X} = 2] + \cdots + 6 \cdot \Pr[\boldsymbol{X} = 6]$$

Expectation of a Random Variable

Let $\boldsymbol{X}$ be the outcome of the roll of a 6-sided die.

$\boldsymbol{X}$ = RandInt(6)

$$
\begin{aligned}
\mathbf{E}[\boldsymbol{X}] &= 1 \cdot \Pr[\boldsymbol{X} = 1] + 2 \cdot \Pr[\boldsymbol{X} = 2] + \cdots + 6 \cdot \Pr[\boldsymbol{X} = 6] \\
&= 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + \cdots + 6 \cdot \frac{1}{6}
\end{aligned}
$$

Expectation of a Random Variable

Let $\boldsymbol{X}$ be the outcome of the roll of a 6-sided die.

$\boldsymbol{X}$ = RandInt(6)

$$\begin{aligned}\mathbf{E}[\boldsymbol{X}] &= 1 \cdot \Pr[\boldsymbol{X} = 1] + 2 \cdot \Pr[\boldsymbol{X} = 2] + \cdots + 6 \cdot \Pr[\boldsymbol{X} = 6] \\ &= 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + \cdots + 6 \cdot \frac{1}{6} \\ &= 3.5\end{aligned}$$

Expectation of a Random Variable

Let $\boldsymbol{X}$ be the outcome of the roll of a 6-sided die.

$\boldsymbol{X}$ = RandInt(6)

$$\begin{aligned}\mathbf{E}[\boldsymbol{X}] &= 1 \cdot \Pr[\boldsymbol{X} = 1] + 2 \cdot \Pr[\boldsymbol{X} = 2] + \cdots + 6 \cdot \Pr[\boldsymbol{X} = 6] \\ &= 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + \cdots + 6 \cdot \frac{1}{6} \\ &= 3.5\end{aligned}$$

What is $\Pr[\boldsymbol{X} = 3.5]$?

Expectation of a Random Variable

Let $\boldsymbol{X}$ be the outcome of the roll of a 6-sided die.

$\boldsymbol{X}$ = RandInt(6)

$$\begin{aligned}\mathbf{E}[\boldsymbol{X}] &= 1 \cdot \Pr[\boldsymbol{X} = 1] + 2 \cdot \Pr[\boldsymbol{X} = 2] + \cdots + 6 \cdot \Pr[\boldsymbol{X} = 6] \\ &= 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + \cdots + 6 \cdot \frac{1}{6} \\ &= 3.5\end{aligned}$$

What is $\Pr[\boldsymbol{X} = 3.5]$?

(Don't always expect the expected!)

Expectation of a Random Variable

Let $\boldsymbol{X}$ = RandInt(6), $\boldsymbol{Y}$ = RandInt(6), $\boldsymbol{Z}$ = RandInt(6)

Let $\boldsymbol{S} = \boldsymbol{X} + \boldsymbol{Y} + \boldsymbol{Z}$

Expectation of a Random Variable

Let $\boldsymbol{X}$ = RandInt(6), $\boldsymbol{Y}$ = RandInt(6), $\boldsymbol{Z}$ = RandInt(6)

Let $\boldsymbol{S} = \boldsymbol{X} + \boldsymbol{Y} + \boldsymbol{Z}$

$$\mathbf{E}[\boldsymbol{S}]$$

Expectation of a Random Variable

Let $\boldsymbol{X}$ = RandInt(6), $\boldsymbol{Y}$ = RandInt(6), $\boldsymbol{Z}$ = RandInt(6)

Let $\boldsymbol{S} = \boldsymbol{X} + \boldsymbol{Y} + \boldsymbol{Z}$

$$\mathbf{E}[\boldsymbol{S}] = 3 \cdot \Pr[\boldsymbol{S} = 3] + 4 \cdot \Pr[\boldsymbol{S} = 4] + \cdots + 18 \cdot \Pr[\boldsymbol{S} = 18]$$

Expectation of a Random Variable

Let $\boldsymbol{X}$ = RandInt(6), $\boldsymbol{Y}$ = RandInt(6), $\boldsymbol{Z}$ = RandInt(6)

Let $\boldsymbol{S} = \boldsymbol{X} + \boldsymbol{Y} + \boldsymbol{Z}$

$$\mathbf{E}[\boldsymbol{S}] = 3 \cdot \Pr[\boldsymbol{S} = 3] + 4 \cdot \Pr[\boldsymbol{S} = 4] + \cdots + 18 \cdot \Pr[\boldsymbol{S} = 18]$$

lot's of arithmetic :-(

Expectation of a Random Variable

Let $\boldsymbol{X}$ = RandInt(6), $\boldsymbol{Y}$ = RandInt(6), $\boldsymbol{Z}$ = RandInt(6)

Let $\boldsymbol{S} = \boldsymbol{X} + \boldsymbol{Y} + \boldsymbol{Z}$

$$\mathbf{E}[\boldsymbol{S}] = 3 \cdot \Pr[\boldsymbol{S} = 3] + 4 \cdot \Pr[\boldsymbol{S} = 4] + \cdots + 18 \cdot \Pr[\boldsymbol{S} = 18]$$

lot's of arithmetic :-(

$$= 10.5$$

Linearity of Expectation example

Linearity of Expectation example

Let $\mathbf{R}_1$ = RandInt(6), $\mathbf{R}_2$ = RandInt(6),

$\mathbf{S} = \mathbf{R}_1 + \mathbf{R}_2$.

Linearity of Expectation example

Let $\mathbf{R}_1$ = RandInt(6), $\mathbf{R}_2$ = RandInt(6),

$\mathbf{S} = \mathbf{R}_1 + \mathbf{R}_2$.

$$\mathbf{E}[\mathbf{S}] = \mathbf{E}[\mathbf{R}_1] + \mathbf{E}[\mathbf{R}_2]$$

Linearity of Expectation example

Let $\mathbf{R}_1$ = RandInt(6), $\mathbf{R}_2$ = RandInt(6),

$\mathbf{S} = \mathbf{R}_1 + \mathbf{R}_2$.

$$\mathbf{E}[\mathbf{S}] = \mathbf{E}[\mathbf{R}_1] + \mathbf{E}[\mathbf{R}_2]$$

$$= 3.5 + 3.5$$

Linearity of Expectation example

Let $\mathbf{R}_1$ = RandInt(6), $\mathbf{R}_2$ = RandInt(6),

$\mathbf{S} = \mathbf{R}_1 + \mathbf{R}_2$.

$$\mathbf{E}[\mathbf{S}] = \mathbf{E}[\mathbf{R}_1] + \mathbf{E}[\mathbf{R}_2]$$

$$= 3.5 + 3.5$$

$$= 7$$

Expectation of an Indicator

Expectation of an Indicator

Fact:

Let A be an event, let **X** be its indicator rand. vbl.

Then $\mathbf{E}[\mathbf{X}] = \mathbf{Pr}[A]$.

Expectation of an Indicator

Fact:

Let A be an event, let **X** be its indicator rand. vbl.

Then **E**[**X**] = **Pr**[A].

Proof:

$$\begin{aligned}\mathbf{E}[\mathbf{X}] &= \sum_{\ell\in\Omega} \mathbf{Pr}[\ell]\cdot\mathbf{X}(\ell)\\ &= \sum_{\ell\in A} \mathbf{Pr}[\ell]\cdot 1 + \sum_{\ell\notin A} \mathbf{Pr}[\ell]\cdot 0\\ &= \sum_{\ell\in A} \mathbf{Pr}[\ell]\\ &= \mathbf{Pr}[A]\end{aligned}$$

Linearity of Expectation

+

Indicators

Linearity of Expectation

+

Indicators

= best friends forever

Linearity of Expectation + Indicators

Linearity of Expectation + Indicators

There are 251 students in a class.

The TAs randomly permute their midterms before handing them back.

Let **X** be the number of students getting their own midterm back.

What is **E**[**X**]?

Let’s try 3 students first

Let's try 3 students first

Midterm they got

Student 1	Student 2	Student 3	Prob	X = # getting own midterm
1	2	3	1/6	3
1	3	2	1/6	1
2	1	3	1/6	1
2	3	1	1/6	0
3	1	2	1/6	0
3	2	1	1/6	1

Let's try 3 students first

Midterm they got

Student 1	Student 2	Student 3	Prob	X = # getting own midterm
1	2	3	1/6	3
1	3	2	1/6	1
2	1	3	1/6	1
2	3	1	1/6	0
3	1	2	1/6	0
3	2	1	1/6	1

$\therefore$ **E[X]** $= (1/6)(3+1+1+0+0+1) = 1$

Now let's do 251 students

Now let's do 251 students

		Um…		

Now let’s do 251 students

Now let's do 251 students

Let A_i be the event that i^{th} student gets own midterm.

Now let's do 251 students

Let A_i be the event that i^{th} student gets own midterm.

Let $\mathbf{X}_i$ be the indicator of A_i.

Now let's do 251 students

Let A_i be the event that i^{th} student gets own midterm.

Let $\mathbf{X}_i$ be the indicator of A_i.

Then $\mathbf{X} =$

Now let's do 251 students

Let A_i be the event that i^{th} student gets own midterm.

Let $\mathbf{X}_i$ be the indicator of A_i.

Then $\mathbf{X} = \mathbf{X}_1 + \mathbf{X}_2 + \cdots + \mathbf{X}_{251}$

Now let's do 251 students

Let A_i be the event that i^{th} student gets own midterm.

Let $\mathbf{X}_i$ be the indicator of A_i.

Then $\mathbf{X} = \mathbf{X}_1 + \mathbf{X}_2 + \cdots + \mathbf{X}_{251}$

Thus $\mathbf{E}[\mathbf{X}] = \mathbf{E}[\mathbf{X}_1] + \mathbf{E}[\mathbf{X}_2] + \cdots + \mathbf{E}[\mathbf{X}_{251}]$

by linearity of expectation

Now let's do 251 students

Let A_i be the event that i^{th} student gets own midterm.

Let $\mathbf{X}_i$ be the indicator of A_i.

Then $\mathbf{X} = \quad \mathbf{X}_1 + \mathbf{X}_2 + \cdots + \mathbf{X}_{251}$

Thus $\mathbf{E}[\mathbf{X}] = \mathbf{E}[\mathbf{X}_1] + \mathbf{E}[\mathbf{X}_2] + \cdots + \mathbf{E}[\mathbf{X}_{251}]$

by linearity of expectation

$\mathbf{E}[\mathbf{X}_i] = \mathbf{Pr}[A_i]$, and $\mathbf{Pr}[A_i] =$

Now let's do 251 students

Let A_i be the event that i^{th} student gets own midterm.

Let $\mathbf{X}_i$ be the indicator of A_i.

Then $\mathbf{X} = \quad \mathbf{X}_1 + \mathbf{X}_2 + \cdots + \mathbf{X}_{251}$

Thus $\mathbf{E}[\mathbf{X}] = \mathbf{E}[\mathbf{X}_1] + \mathbf{E}[\mathbf{X}_2] + \cdots + \mathbf{E}[\mathbf{X}_{251}]$

by linearity of expectation

$\mathbf{E}[\mathbf{X}_i] = \mathbf{Pr}[A_i]$, and $\mathbf{Pr}[A_i] = \quad 1/251$ for each i.

Now let's do 251 students

Let A_i be the event that i^{th} student gets own midterm.

Let $\mathbf{X}_i$ be the indicator of A_i.

Then $\mathbf{X} = \mathbf{X}_1 + \mathbf{X}_2 + \cdots + \mathbf{X}_{251}$

Thus $\mathbf{E}[\mathbf{X}] = \mathbf{E}[\mathbf{X}_1] + \mathbf{E}[\mathbf{X}_2] + \cdots + \mathbf{E}[\mathbf{X}_{251}]$

by linearity of expectation

$\mathbf{E}[\mathbf{X}_i] = \mathbf{Pr}[A_i]$, and $\mathbf{Pr}[A_i] = 1/251$ for each i.

$\therefore \mathbf{E}[\mathbf{X}] = 251 \cdot (1/251) = 1$

Questions

Can we go back to answering more questions in lecture?

I am still confused what is NP. After trying to get an intuition of its definition, I am still confused here.

I gave ChatGPT a piece a piece of code and it correctly told me that it wouldn't halt. I gave it a different piece of code and it correctly told me that it would. Does this mean ChatGPT can solve some instances of the halting problem?

What does mean in the context of computability (ie this kind of seems to contradict that HALTS is undecidable so is AI changing the limits of what is computable by being "computationally stronger" than TMs?)?

What are some problems that were thought to be in NP but was actually in P?

What are ways that people attempt to prove P = NP? Is it just trying to develop polynomial-time algorithms for NP-complete problems, or are there other methods?

From what was told in lecture, it seems like we are far from having the capabilities of answering the question of whether P = NP. If this is the case, is the question still worth pursuing? How much progress has been made in answering the question?

What other large complexity classes are there?

Why is clique no complete when clique 251 is in P?
Isn’t clique 251 just clique with k = 251?

why do we need the first and last vertices to be unmatched in an augmenting path? Is the augmenting path only for characterization of maximum matching?

Why is Gale-Shapley even used if there are algorithms that can output "more even/fair" stable matchings, such that Y is not always matched with its worst valid partner?

Do you think humans are deterministic TMs or not?
Can emotions be completely modeled?

Some More Random Variables

Example

Example

```
X ← RandInt(2)
Y ← RandInt(X+1)
```

Example

$$\mathbf{X} \leftarrow \texttt{RandInt}(2)$$

$$\mathbf{Y} \leftarrow \texttt{RandInt}(\mathbf{X}+1)$$

Question: What is **E[XY]**?

Example

$$\mathbf{X} \leftarrow \texttt{RandInt(2)}$$
$$\mathbf{Y} \leftarrow \texttt{RandInt(}\mathbf{X}\texttt{+1)}$$

Question: What is **E[XY]**?

$$
\begin{aligned}
= \ & \mathbf{Pr}[\mathbf{X}=1 \cap \mathbf{Y}=1]\cdot 1\cdot 1 \\
+ & \mathbf{Pr}[\mathbf{X}=1 \cap \mathbf{Y}=2]\cdot 1\cdot 2 \\
+ & \mathbf{Pr}[\mathbf{X}=1 \cap \mathbf{Y}=3]\cdot 1\cdot 3 \\
+ & \mathbf{Pr}[\mathbf{X}=2 \cap \mathbf{Y}=1]\cdot 2\cdot 1 \\
+ & \mathbf{Pr}[\mathbf{X}=2 \cap \mathbf{Y}=2]\cdot 2\cdot 2 \\
+ & \mathbf{Pr}[\mathbf{X}=2 \cap \mathbf{Y}=3]\cdot 2\cdot 3
\end{aligned}
$$

Example

$$\mathbf{X} \leftarrow \texttt{RandInt(2)}$$

$$\mathbf{Y} \leftarrow \texttt{RandInt(}\mathbf{X}\texttt{+1)}$$

Question: What is **E[XY]**?

$$
\begin{aligned}
&= \mathbf{Pr}[\mathbf{X}=1 \cap \mathbf{Y}=1]\cdot 1\cdot 1 && = (1/4)\cdot 1\cdot 1\\
&+\mathbf{Pr}[\mathbf{X}=1 \cap \mathbf{Y}=2]\cdot 1\cdot 2 && +(1/4)\cdot 1\cdot 2\\
&+\mathbf{Pr}[\mathbf{X}=1 \cap \mathbf{Y}=3]\cdot 1\cdot 3 && +\ (0)\cdot 1\cdot 3\\
&+\mathbf{Pr}[\mathbf{X}=2 \cap \mathbf{Y}=1]\cdot 2\cdot 1 && +(1/6)\cdot 2\cdot 1\\
&+\mathbf{Pr}[\mathbf{X}=2 \cap \mathbf{Y}=2]\cdot 2\cdot 2 && +(1/6)\cdot 2\cdot 2\\
&+\mathbf{Pr}[\mathbf{X}=2 \cap \mathbf{Y}=3]\cdot 2\cdot 3 && +(1/6)\cdot 2\cdot 3
\end{aligned}
$$

Example

$$\mathbf{X} \leftarrow \texttt{RandInt(2)}$$

$$\mathbf{Y} \leftarrow \texttt{RandInt(}\mathbf{X}\texttt{+1)}$$

Question: What is **E[XY]**?

$$
\begin{aligned}
= \ & \mathbf{Pr}[\mathbf{X}=1 \cap \mathbf{Y}=1]\cdot 1\cdot 1 & = \ & (1/4)\cdot 1\cdot 1 \\
+ & \mathbf{Pr}[\mathbf{X}=1 \cap \mathbf{Y}=2]\cdot 1\cdot 2 & + & (1/4)\cdot 1\cdot 2 \\
+ & \mathbf{Pr}[\mathbf{X}=1 \cap \mathbf{Y}=3]\cdot 1\cdot 3 & + & \ (0)\cdot 1\cdot 3 \\
+ & \mathbf{Pr}[\mathbf{X}=2 \cap \mathbf{Y}=1]\cdot 2\cdot 1 & + & (1/6)\cdot 2\cdot 1 \\
+ & \mathbf{Pr}[\mathbf{X}=2 \cap \mathbf{Y}=2]\cdot 2\cdot 2 & + & (1/6)\cdot 2\cdot 2 \\
+ & \mathbf{Pr}[\mathbf{X}=2 \cap \mathbf{Y}=3]\cdot 2\cdot 3 & + & (1/6)\cdot 2\cdot 3
\end{aligned}
$$

$=11/4$

Example

```
X ← RandInt(2)
Y ← RandInt(X+1)
```

E[XY]

Example

$$\mathbf{X} \leftarrow \mathtt{RandInt(2)}$$

$$\mathbf{Y} \leftarrow \mathtt{RandInt(}\mathbf{X}\mathtt{+1)}$$

$$\mathbf{E[XY]} \quad = 11/4$$

Example

```
X ← RandInt(2)
Y ← RandInt(X+1)
```

$E[XY] = 11/4$

$E[X] =$

Example

```
X ← RandInt(2)
Y ← RandInt(X+1)
```

E[XY] = 11/4

E[X] = 3/2

Example

```
X ← RandInt(2)
Y ← RandInt(X+1)
```

$E[XY] = 11/4$

$E[X] = 3/2$

$E[Y] =$

Example

$\mathbf{X} \leftarrow \texttt{RandInt(2)}$

$\mathbf{Y} \leftarrow \texttt{RandInt(}\mathbf{X}\texttt{+1)}$

E[XY] = 11/4

E[X] = 3/2

E[Y] = 7/4 (exercise)

Example

$\mathbf{X} \leftarrow$ `RandInt(2)`

$\mathbf{Y} \leftarrow$ `RandInt(X+1)`

$\mathbf{E[XY]} = 11/4$

$\mathbf{E[X]} = 3/2$

$\mathbf{E[Y]} = 7/4$ (exercise)

Notice: $\mathbf{E[XY]} \neq \mathbf{E[X]}\,\mathbf{E[Y]}$ in general!

But…

If **X** and **Y** are independent

then **E**[**XY**] = **E**[**X**] **E**[**Y**].

If **X** and **Y** are independent

then **E[XY]** = **E[X] E[Y]**.

Proof:

If **X** and **Y** are independent then **E[XY] = E[X] E[Y]**.

Proof:

$$\mathbf{E}[\mathbf{XY}] = \sum_{\substack{u \in \text{range}(\mathbf{X}) \\ v \in \text{range}(\mathbf{Y})}} \mathbf{Pr}[\mathbf{X} = u \cap \mathbf{Y} = v] \cdot uv$$

$$= \sum_{u,v} \mathbf{Pr}[\mathbf{X} = u]\mathbf{Pr}[\mathbf{Y} = v] \cdot uv \quad \text{(independence!)}$$

$$= \sum_{u,v} p_{\mathbf{X}}(u)u \cdot p_{\mathbf{Y}}(v)v$$

$$= \left(\sum_{u} p_{\mathbf{X}}(u)u\right) \cdot \left(\sum_{v} p_{\mathbf{Y}}(v)v\right)$$

$$= \mathbf{E}[\mathbf{X}]\mathbf{E}[\mathbf{Y}]$$

Most Useful Equality in Probability Theory:

Most Useful Equality in Probability Theory:

Linearity of Expectation

Linearity of Expectation

$$\mathbf{E}[\boldsymbol{X} + \boldsymbol{Y}] = \mathbf{E}[\boldsymbol{X}] + \mathbf{E}[\boldsymbol{Y}]$$

Linearity of Expectation

Let $\boldsymbol{X}$ = RandInt(6), $\boldsymbol{Y}$ = RandInt(6), $\boldsymbol{Z}$ = RandInt(6)

Let $\boldsymbol{S} = \boldsymbol{X} + \boldsymbol{Y} + \boldsymbol{Z}$

Linearity of Expectation

Let $\boldsymbol{X}$ = RandInt(6), $\boldsymbol{Y}$ = RandInt(6), $\boldsymbol{Z}$ = RandInt(6)

Let $\boldsymbol{S} = \boldsymbol{X} + \boldsymbol{Y} + \boldsymbol{Z}$

$\mathbf{E}[\boldsymbol{S}]$

Linearity of Expectation

Let $\boldsymbol{X}$ = RandInt(6), $\boldsymbol{Y}$ = RandInt(6), $\boldsymbol{Z}$ = RandInt(6)

Let $\boldsymbol{S} = \boldsymbol{X} + \boldsymbol{Y} + \boldsymbol{Z}$

$$\mathbf{E}[\boldsymbol{S}] = \mathbf{E}[\boldsymbol{X} + \boldsymbol{Y} + \boldsymbol{Z}]$$

Linearity of Expectation

Let $\boldsymbol{X}$ = RandInt(6), $\boldsymbol{Y}$ = RandInt(6), $\boldsymbol{Z}$ = RandInt(6)

Let $\boldsymbol{S} = \boldsymbol{X} + \boldsymbol{Y} + \boldsymbol{Z}$

$$\begin{aligned}\mathbf{E}[\boldsymbol{S}] &= \mathbf{E}[\boldsymbol{X} + \boldsymbol{Y} + \boldsymbol{Z}] \\ &= \mathbf{E}[\boldsymbol{X}] + \mathbf{E}[\boldsymbol{Y} + \boldsymbol{Z}]\end{aligned}$$

Linearity of Expectation

Let $\boldsymbol{X}$ = RandInt(6), $\boldsymbol{Y}$ = RandInt(6), $\boldsymbol{Z}$ = RandInt(6)

Let $\boldsymbol{S} = \boldsymbol{X} + \boldsymbol{Y} + \boldsymbol{Z}$

$$
\begin{aligned}
\mathbf{E}[\boldsymbol{S}] &= \mathbf{E}[\boldsymbol{X} + \boldsymbol{Y} + \boldsymbol{Z}] \\
&= \mathbf{E}[\boldsymbol{X}] + \mathbf{E}[\boldsymbol{Y} + \boldsymbol{Z}] \\
&= \mathbf{E}[\boldsymbol{X}] + \mathbf{E}[\boldsymbol{Y}] + \mathbf{E}[\boldsymbol{Z}]
\end{aligned}
$$

Linearity of Expectation

Let $\boldsymbol{X}$ = RandInt(6), $\boldsymbol{Y}$ = RandInt(6), $\boldsymbol{Z}$ = RandInt(6)

Let $\boldsymbol{S} = \boldsymbol{X} + \boldsymbol{Y} + \boldsymbol{Z}$

$$\begin{aligned}
\mathbf{E}[\boldsymbol{S}] &= \mathbf{E}[\boldsymbol{X} + \boldsymbol{Y} + \boldsymbol{Z}] \\
&= \mathbf{E}[\boldsymbol{X}] + \mathbf{E}[\boldsymbol{Y} + \boldsymbol{Z}] \\
&= \mathbf{E}[\boldsymbol{X}] + \mathbf{E}[\boldsymbol{Y}] + \mathbf{E}[\boldsymbol{Z}] \\
&= 3.5 + 3.5 + 3.5
\end{aligned}$$

Linearity of Expectation

Let $\boldsymbol{X}$ = RandInt(6), $\boldsymbol{Y}$ = RandInt(6), $\boldsymbol{Z}$ = RandInt(6)

Let $\boldsymbol{S} = \boldsymbol{X} + \boldsymbol{Y} + \boldsymbol{Z}$

$$\begin{aligned}
\mathbf{E}[\boldsymbol{S}] &= \mathbf{E}[\boldsymbol{X} + \boldsymbol{Y} + \boldsymbol{Z}] \\
&= \mathbf{E}[\boldsymbol{X}] + \mathbf{E}[\boldsymbol{Y} + \boldsymbol{Z}] \\
&= \mathbf{E}[\boldsymbol{X}] + \mathbf{E}[\boldsymbol{Y}] + \mathbf{E}[\boldsymbol{Z}] \\
&= 3.5 + 3.5 + 3.5 \\
&= 10.5
\end{aligned}$$

One of the Most Useful Inequalities in Probability Theory:

One of the Most Useful Inequalities in Probability Theory:

Markov’s Inequality

Markov's Inequality

Markov's Inequality

Markov's Inequality

A non-negative random variable $\boldsymbol{X}$ is rarely much bigger than its expectation $\mathbf{E}[\boldsymbol{X}]$.

Markov's Inequality

A non-negative random variable $\boldsymbol{X}$ is rarely much bigger than its expectation $\mathbf{E}[\boldsymbol{X}]$.

Theorem:

Let $\boldsymbol{X}$ be a random variable that is always non-negative.

Then for any $c \geq 1$,

Markov's Inequality

A non-negative random variable $\boldsymbol{X}$ is rarely much bigger than its expectation $\mathbf{E}[\boldsymbol{X}]$.

Theorem:

Let $\boldsymbol{X}$ be a random variable that is always non-negative.

Then for any $c \geq 1$,

$$\mathbf{Pr}[\boldsymbol{X} \geq c \cdot \mathbf{E}[\boldsymbol{X}]] \leq \frac{1}{c} \ .$$

Most Common 3 Random Variables

Bernoulli Random Variable

Bernoulli Random Variable

Math: $X \sim \mathrm{Bernoulli}(p)$

Bernoulli Random Variable

<u>Math:</u> $\boldsymbol{X} \sim \mathrm{Bernoulli}(p)$

"$\boldsymbol{X}$ is a Bernoulli random variable with success probability p."

Bernoulli Random Variable

<u>Math:</u> $\boldsymbol{X} \sim \mathrm{Bernoulli}(p)$

"$\boldsymbol{X}$ is a Bernoulli random variable with success probability p."

<u>Intuition:</u> Flip a p-biased coin.

Bernoulli Random Variable

Math: $\boldsymbol{X} \sim \mathrm{Bernoulli}(p)$

"$\boldsymbol{X}$ is a Bernoulli random variable with success probability p."

Intuition: Flip a p-biased coin.

Code:

```
X = Bernoulli(p)
```

Bernoulli Random Variable

Math: $X \sim \mathrm{Bernoulli}(p)$

"X is a Bernoulli random variable with success probability p."

Intuition: Flip a p-biased coin.

Code:

```
X = Bernoulli(p)
```

Properties:

Bernoulli Random Variable

Math: $\boldsymbol{X} \sim \text{Bernoulli}(p)$

"$\boldsymbol{X}$ is a Bernoulli random variable with success probability p."

Intuition: Flip a p-biased coin.

Code:

```
X = Bernoulli(p)
```

Properties: $\text{range}(\boldsymbol{X}) = \{0, 1\}$

Bernoulli Random Variable

Math: $\boldsymbol{X} \sim \text{Bernoulli}(p)$

"$\boldsymbol{X}$ is a Bernoulli random variable with success probability p."

Intuition: Flip a p-biased coin.

Code:

```
X = Bernoulli(p)
```

Properties: $\text{range}(\boldsymbol{X}) = \{0, 1\}$

$\Pr[\boldsymbol{X} = 1] = p$

Bernoulli Random Variable

Math: $\boldsymbol{X} \sim \mathrm{Bernoulli}(p)$

"$\boldsymbol{X}$ is a Bernoulli random variable with success probability p."

Intuition: Flip a p-biased coin.

Code:

$$\boxed{\mathrm{X} = \mathrm{Bernoulli}(p)}$$

Properties:

$$\mathrm{range}(\boldsymbol{X}) = \{0, 1\}$$

$$\Pr[\boldsymbol{X} = 1] = p$$

$$\Pr[\boldsymbol{X} = 0] = 1 - p$$

Bernoulli Random Variable

Math: $\boldsymbol{X} \sim \text{Bernoulli}(p)$

"$\boldsymbol{X}$ is a Bernoulli random variable with success probability p."

Intuition: Flip a p-biased coin.

Code:

```
X = Bernoulli(p)
```

Properties: $\text{range}(\boldsymbol{X}) = \{0, 1\}$

$$\Pr[\boldsymbol{X} = 1] = p$$

$$\Pr[\boldsymbol{X} = 0] = 1 - p$$

$$\mathbf{E}[\boldsymbol{X}] = p$$

Binomial Random Variable

Binomial Random Variable

<u>Math:</u> $\boldsymbol{X} \sim \mathrm{Binomial}(n, p)$

Binomial Random Variable

<u>**Math:**</u> $\boldsymbol{X} \sim \mathrm{Binomial}(n, p)$

<u>**Intuition:**</u> Flip n p-biased coins. Interested in number heads/successes.

Binomial Random Variable

Math: $X \sim \text{Binomial}(n, p)$

Intuition: Flip n p-biased coins. Interested in number heads/successes.

Code:

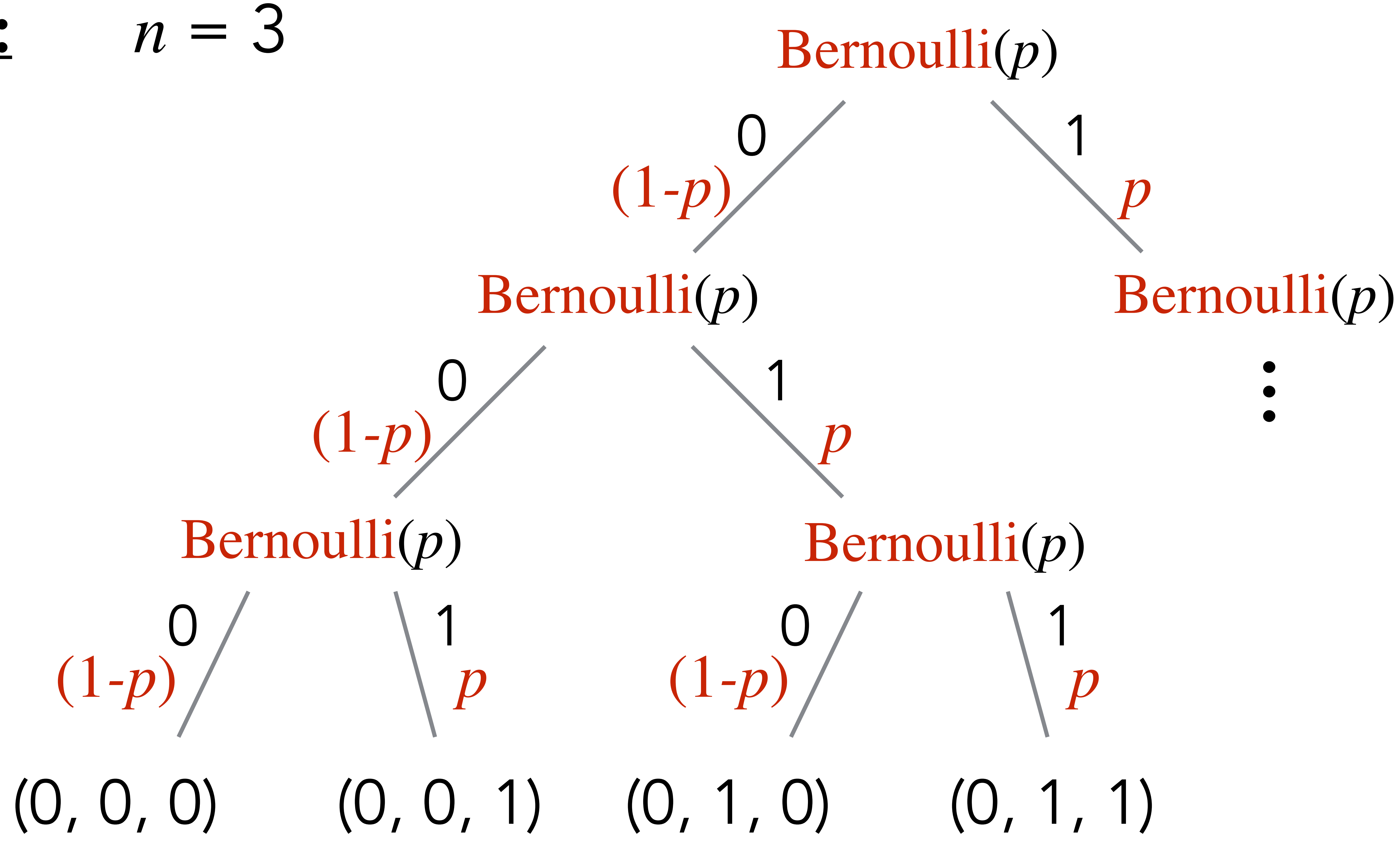
```
X = 0
repeat n times:
    X += Bernoulli(p)
```

Binomial Random Variable

Tree:

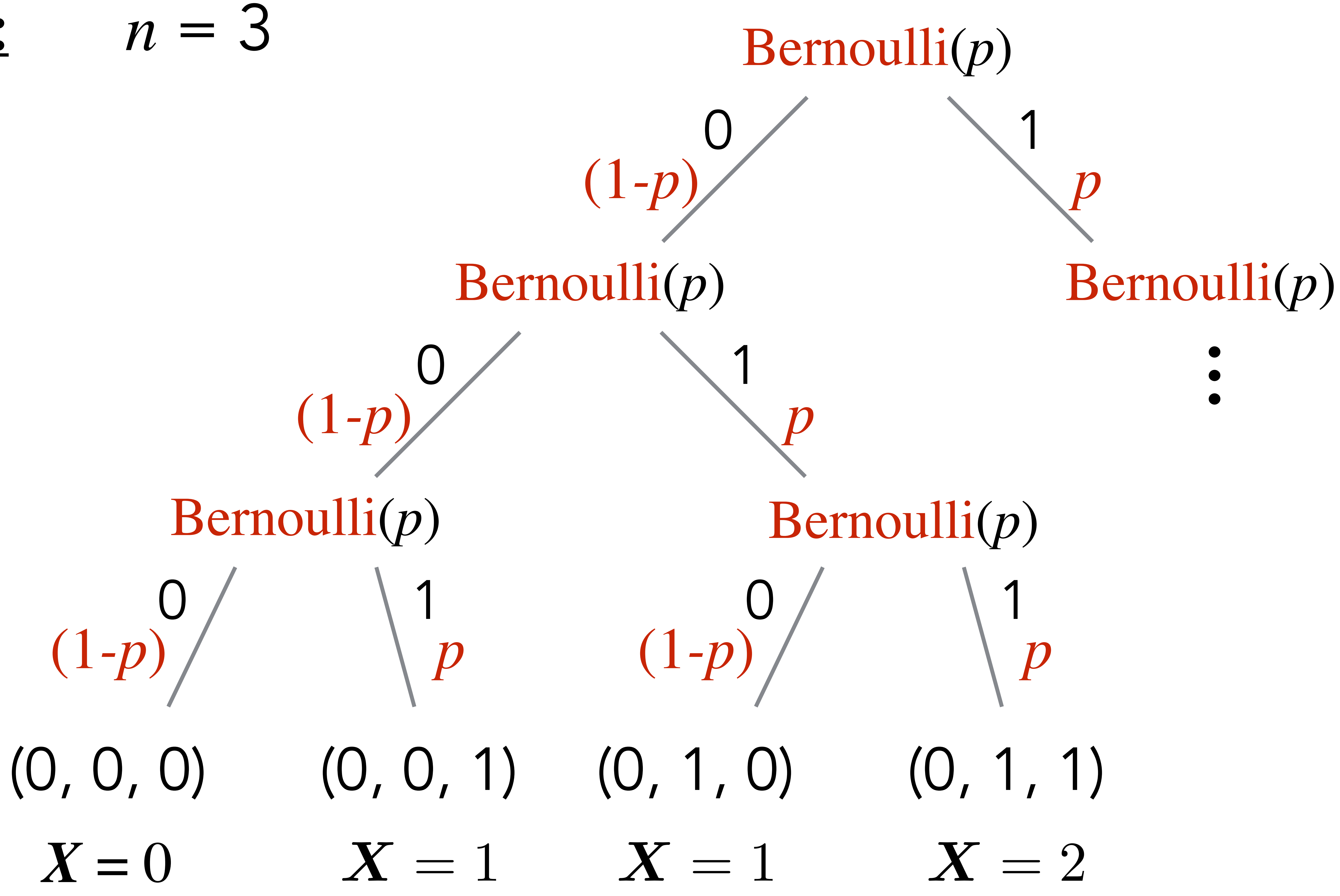
Binomial Random Variable

Tree: $n = 3$



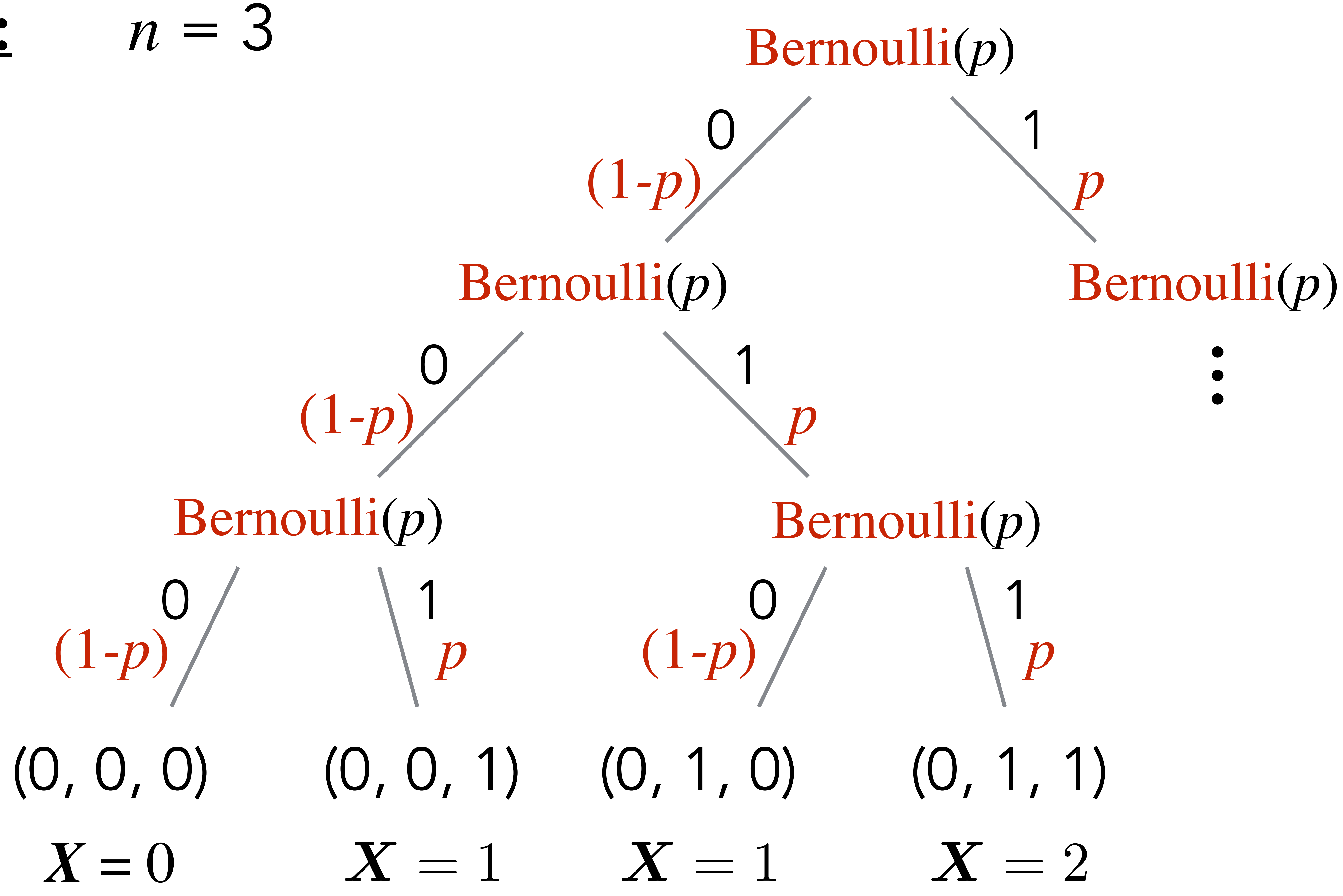
Binomial Random Variable

Tree: $n = 3$



Binomial Random Variable

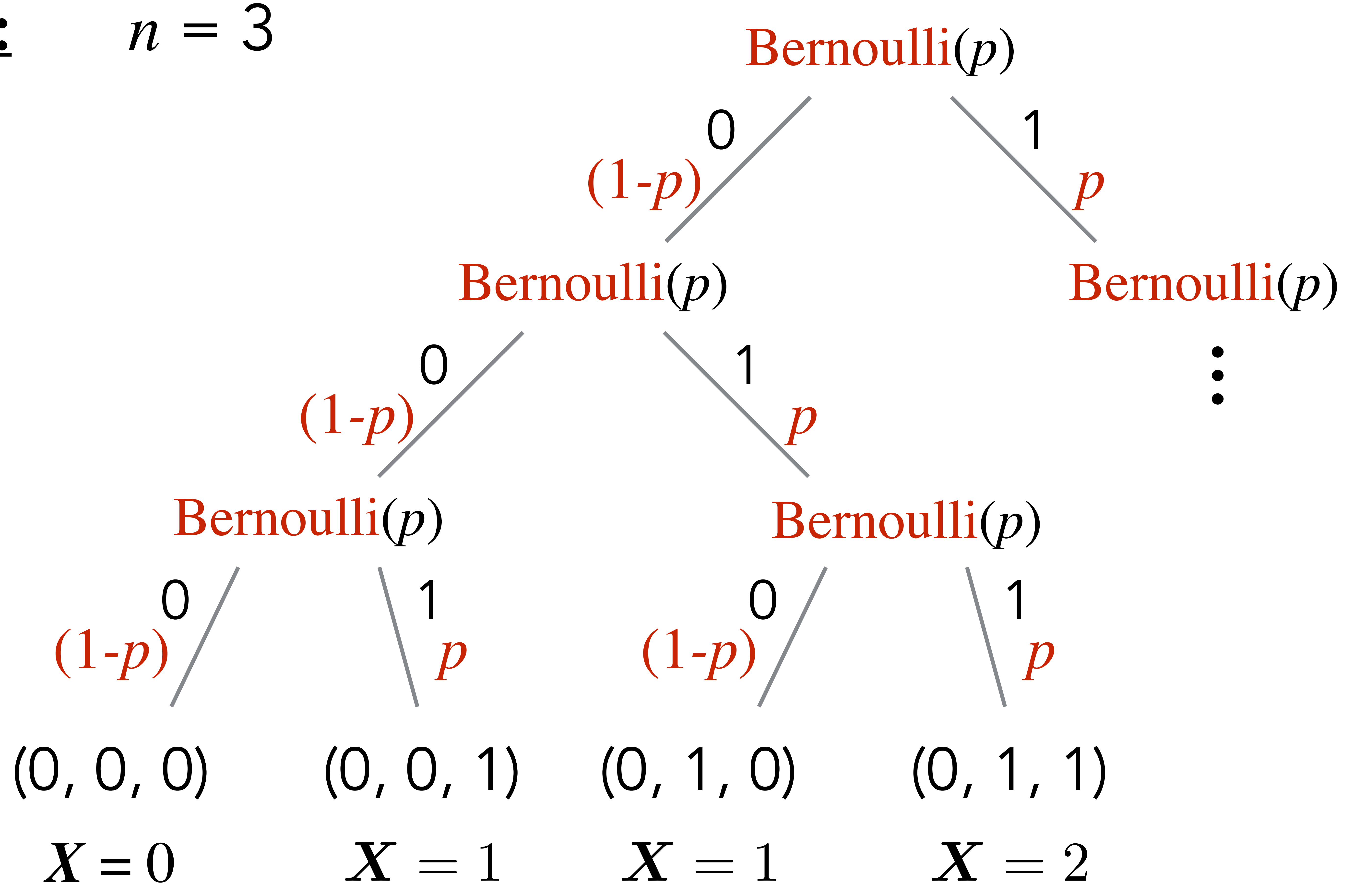
Tree: $n = 3$



Properties:

Binomial Random Variable

Tree: $n = 3$

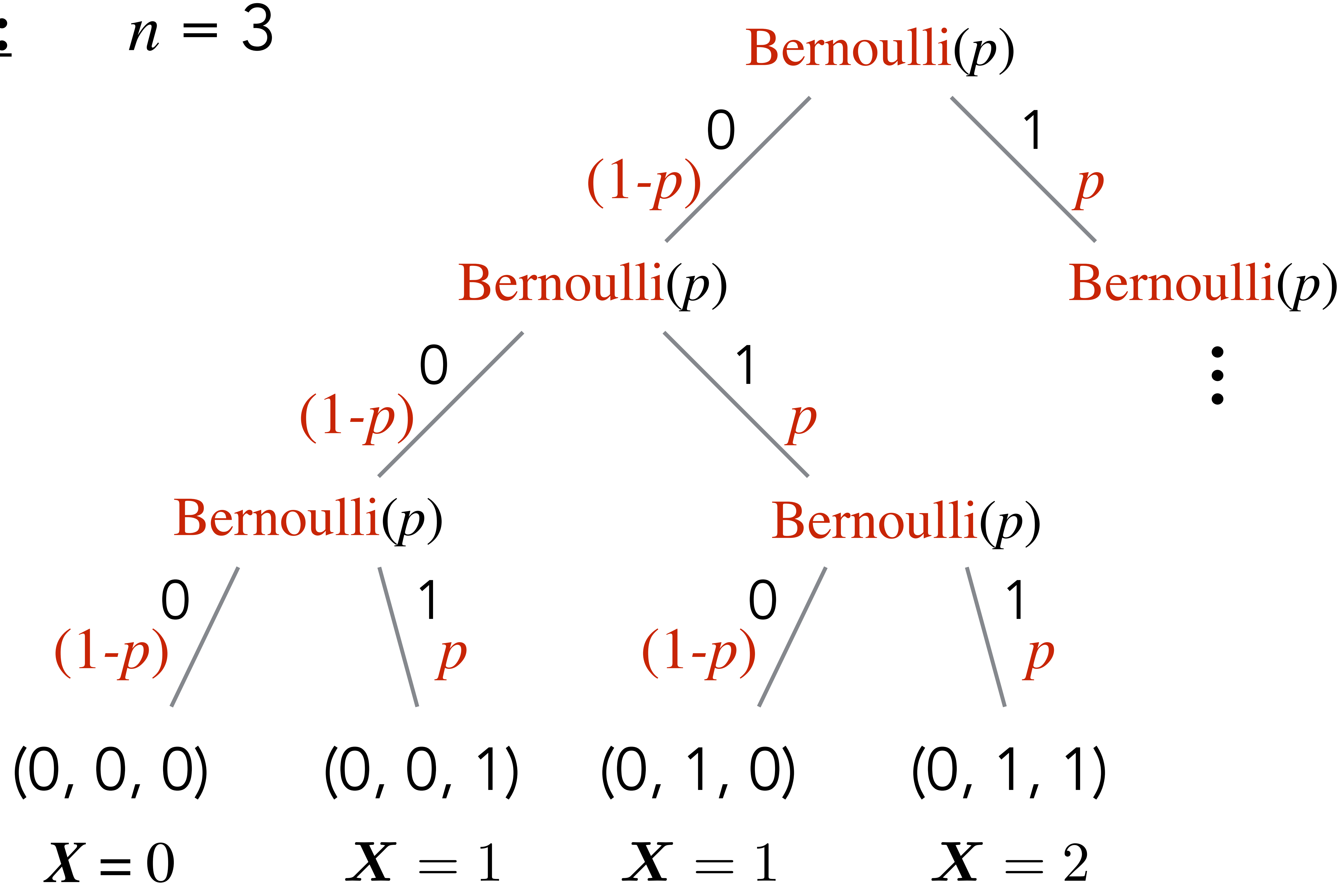


Properties:

$$\text{range}(\boldsymbol{X}) = \{0, 1, 2, \ldots, n\}$$

Binomial Random Variable

Tree: $n = 3$



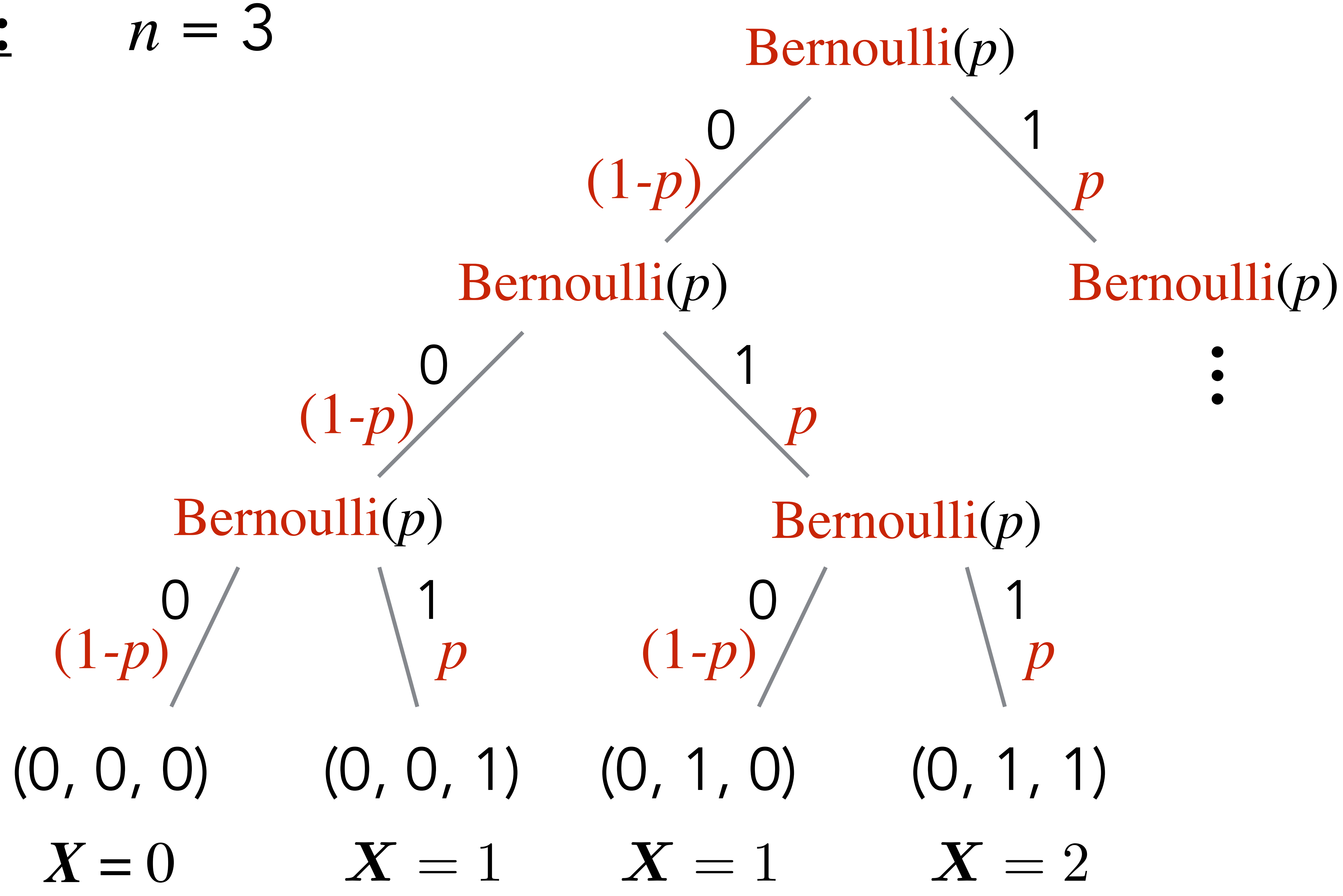
Properties:

$$\text{range}(\boldsymbol{X}) = \{0, 1, 2, \ldots, n\}$$

$$\Pr[\boldsymbol{X} = i] = \binom{n}{i} p^i (1-p)^{n-i}$$

Binomial Random Variable

Tree: $n = 3$



Properties:

$$\text{range}(\boldsymbol{X}) = \{0, 1, 2, \ldots, n\}$$

$$\Pr[\boldsymbol{X} = i] = \binom{n}{i} p^i (1-p)^{n-i}$$

$$\mathbf{E}[\boldsymbol{X}] = np$$

Geometric Random Variable

Geometric Random Variable

<u>**Math:**</u> $X \sim \text{Geometric}(p)$

Geometric Random Variable

Math: $X \sim \text{Geometric}(p)$

Intuition: Number of p-biased coin flips until we see heads/success for the first time.

Geometric Random Variable

Math: $X \sim \text{Geometric}(p)$

Intuition: Number of p-biased coin flips until we see heads/success for the first time.

Code:

```
X = 1
while Bernoulli(p) == 0:
    X += 1
```

Geometric Random Variable

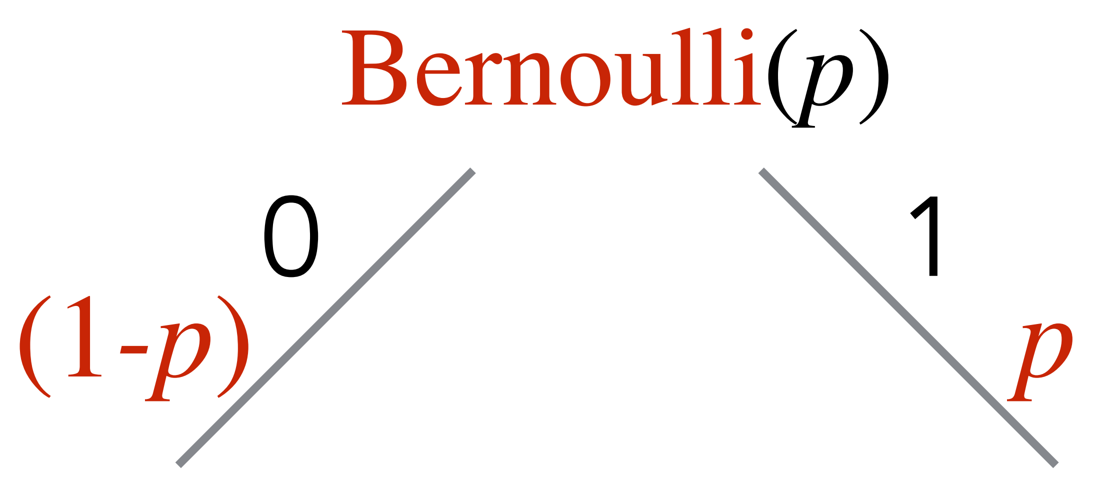
Tree:

Geometric Random Variable

Tree: Bernoulli(p)

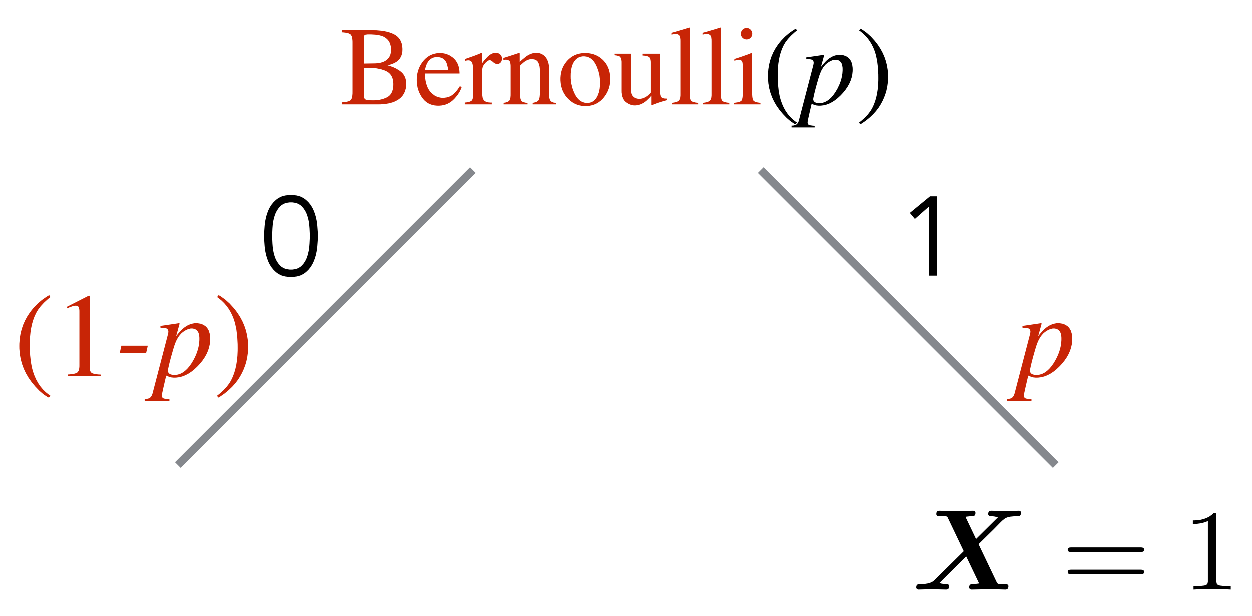
Geometric Random Variable

Tree:



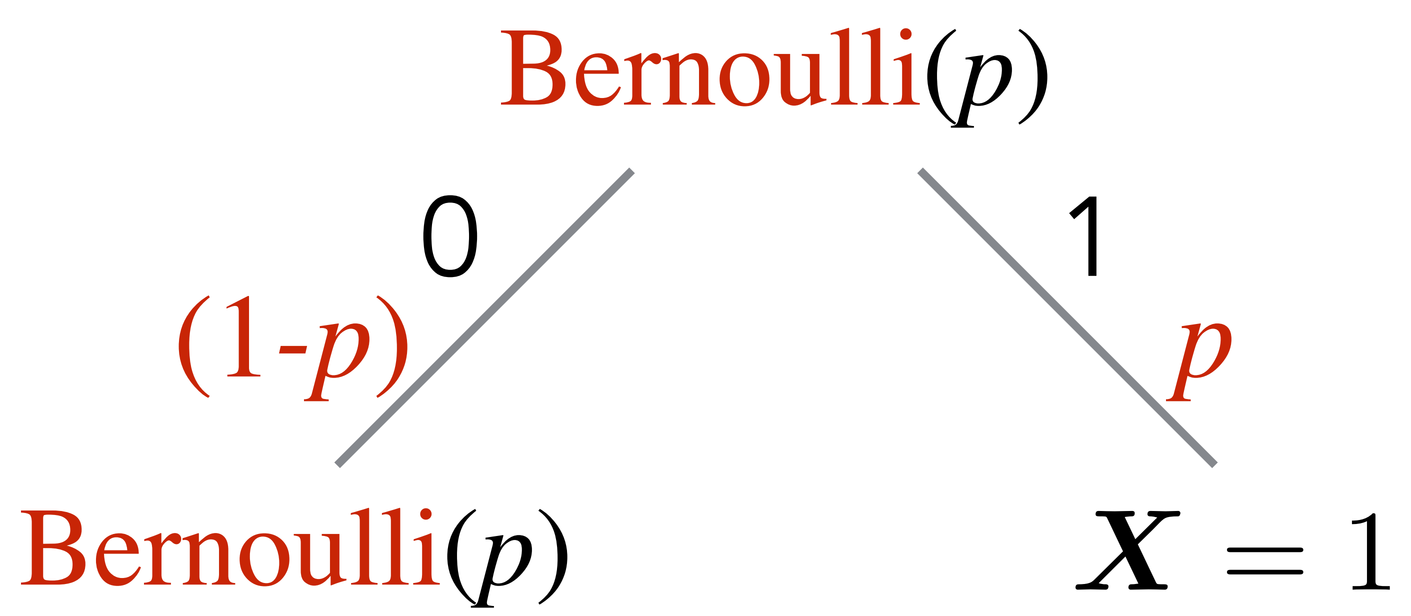
Geometric Random Variable

Tree:



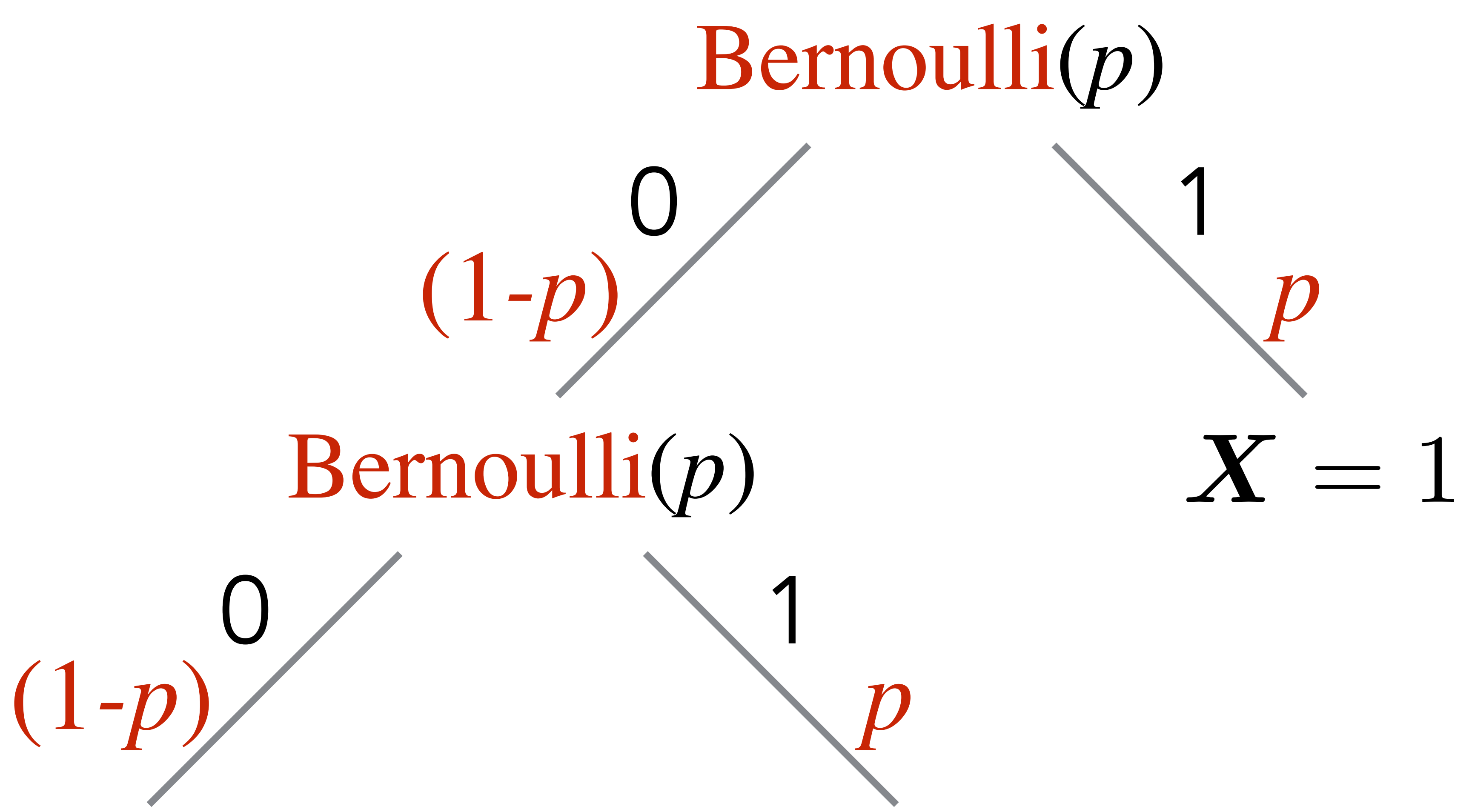
Geometric Random Variable

Tree:



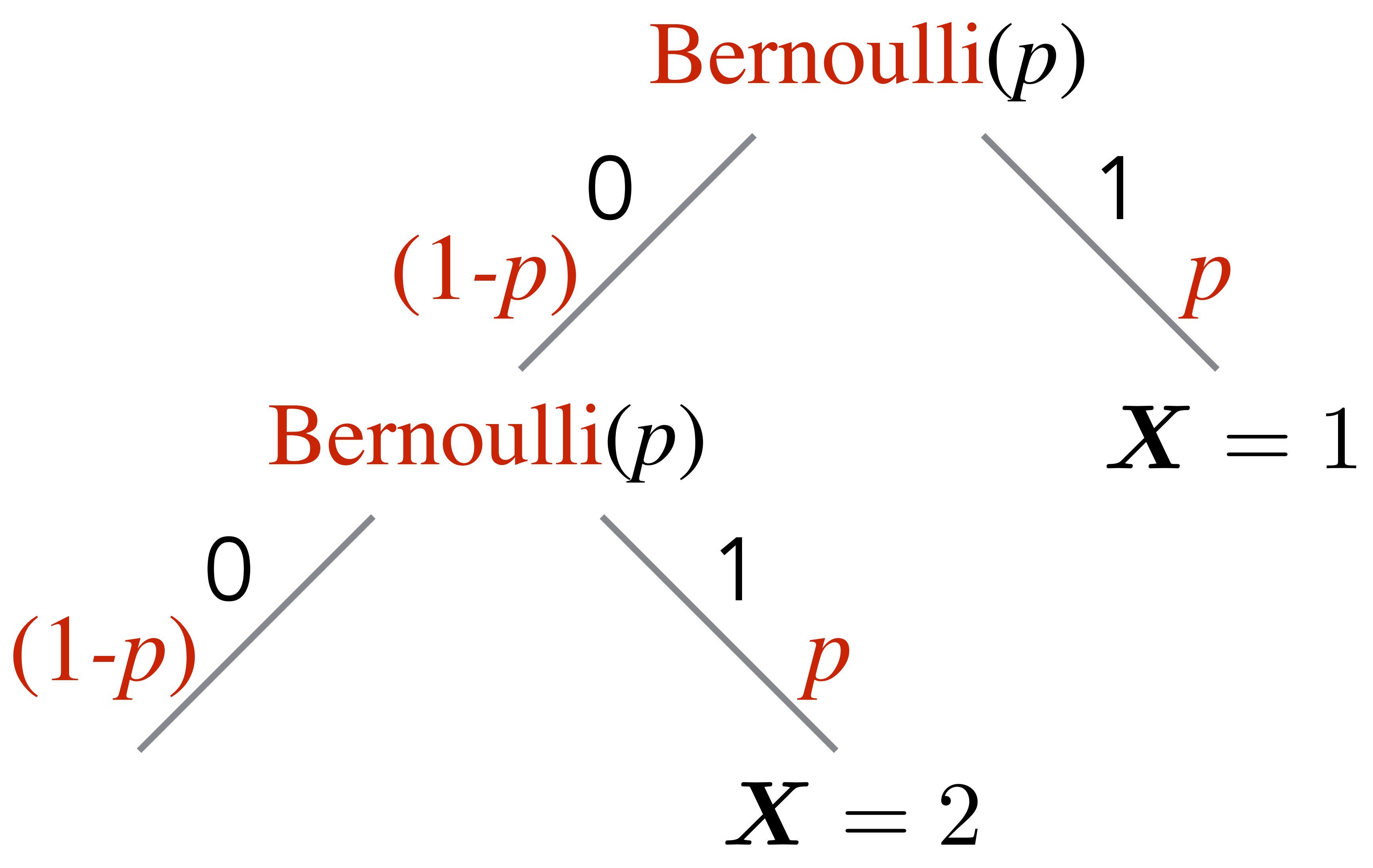
Geometric Random Variable

Tree:



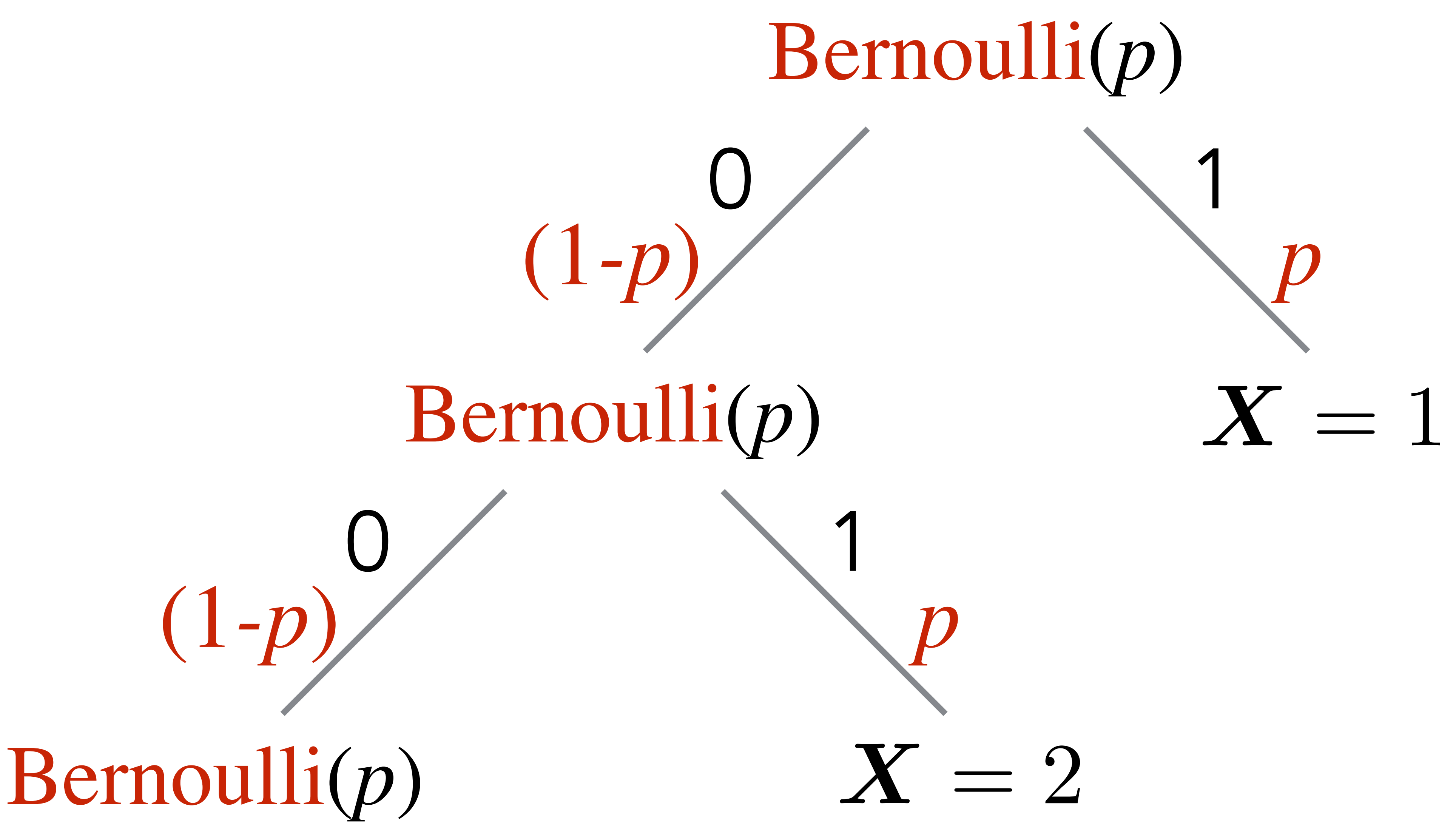
Geometric Random Variable

Tree:



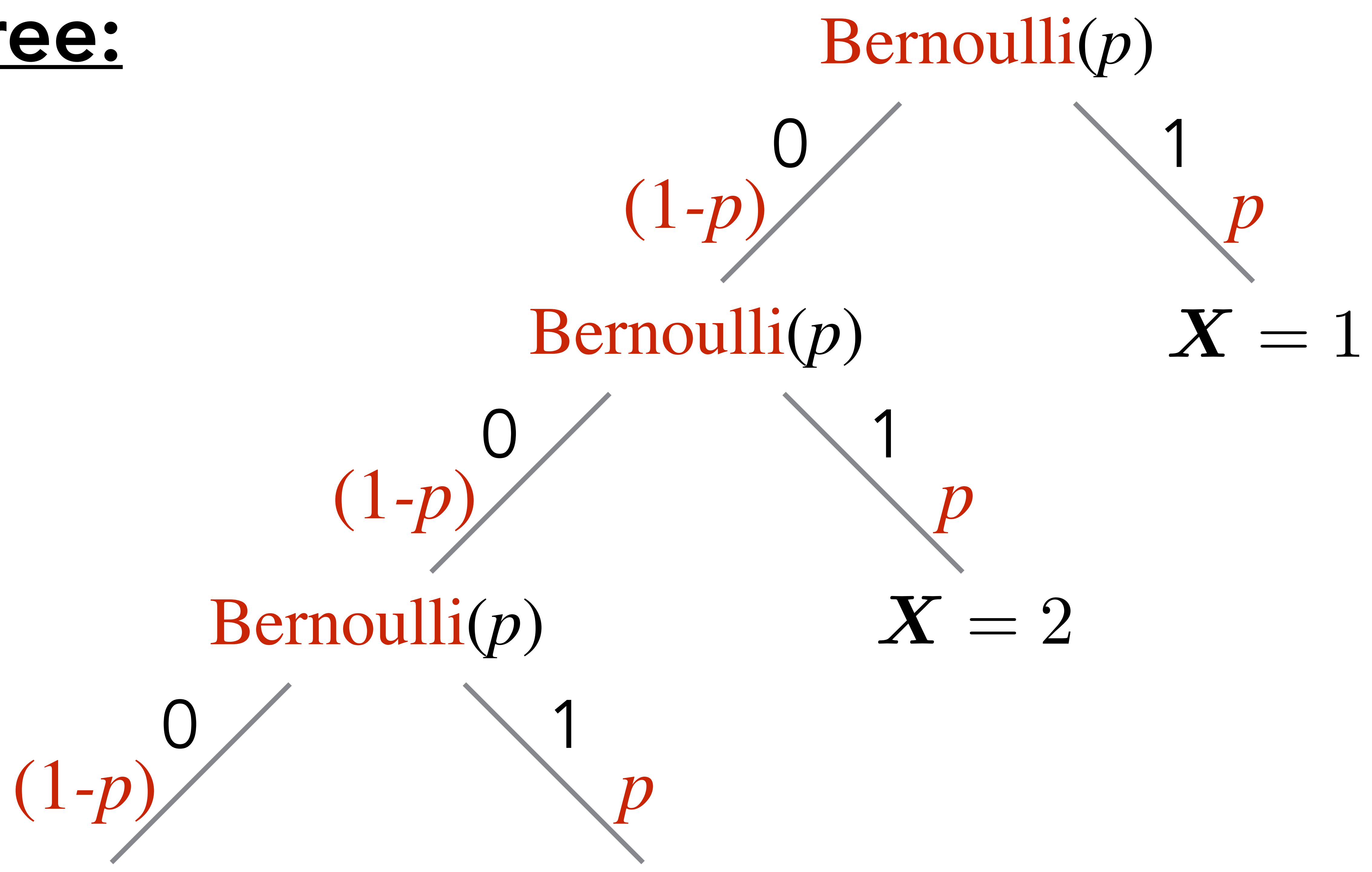
Geometric Random Variable

Tree:



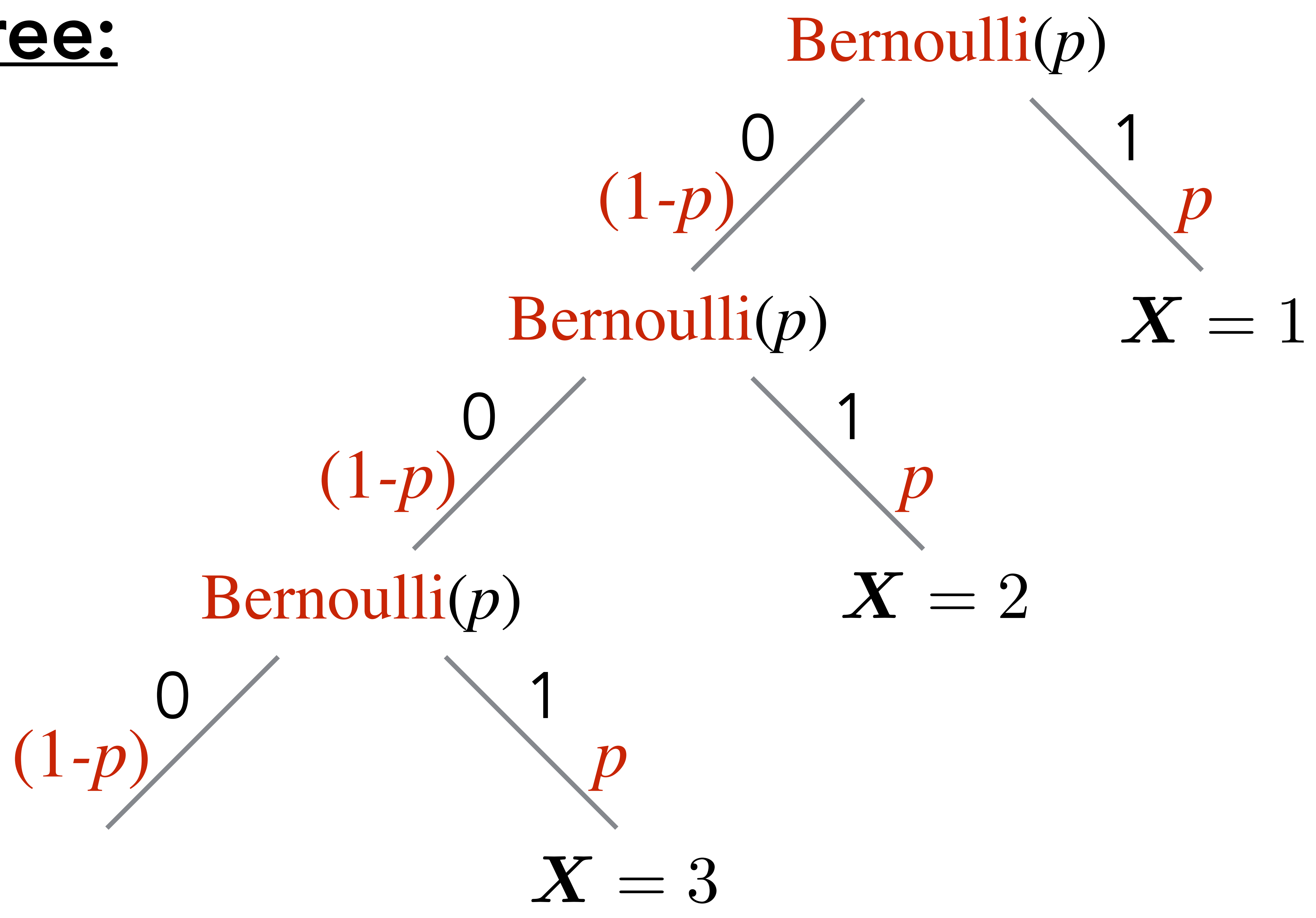
Geometric Random Variable

Tree:



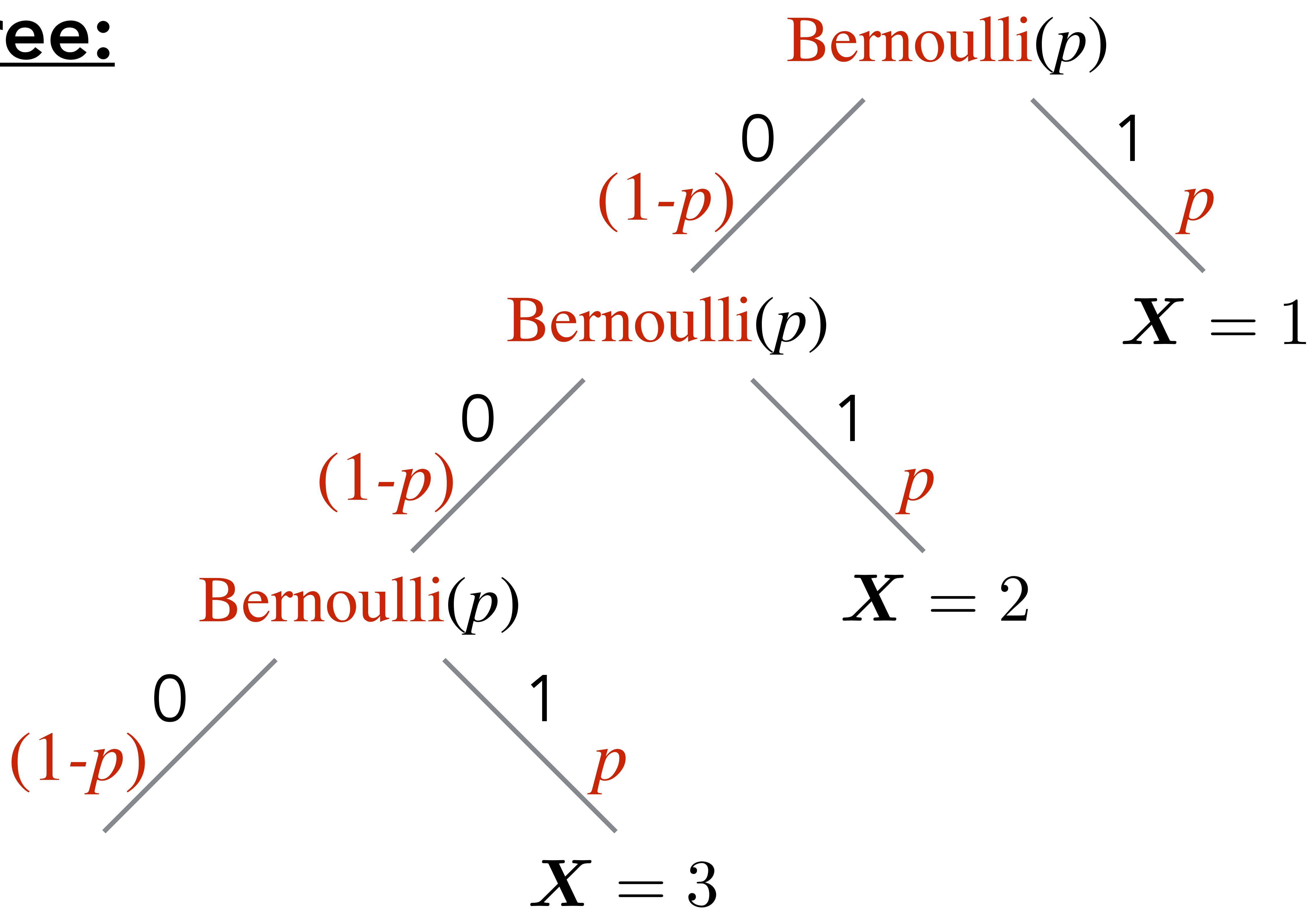
Geometric Random Variable

Tree:



Geometric Random Variable

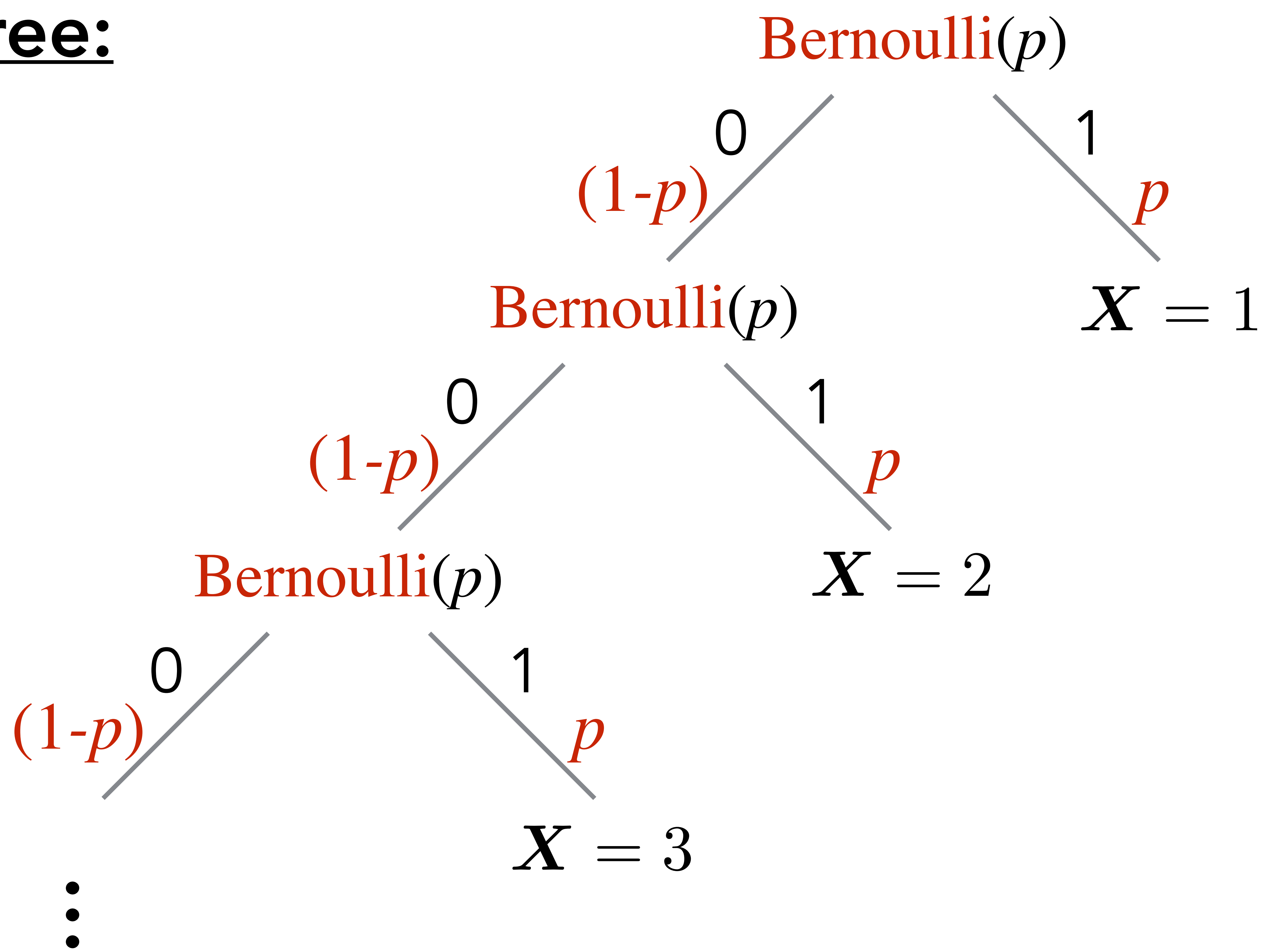
Tree:



Properties:

Geometric Random Variable

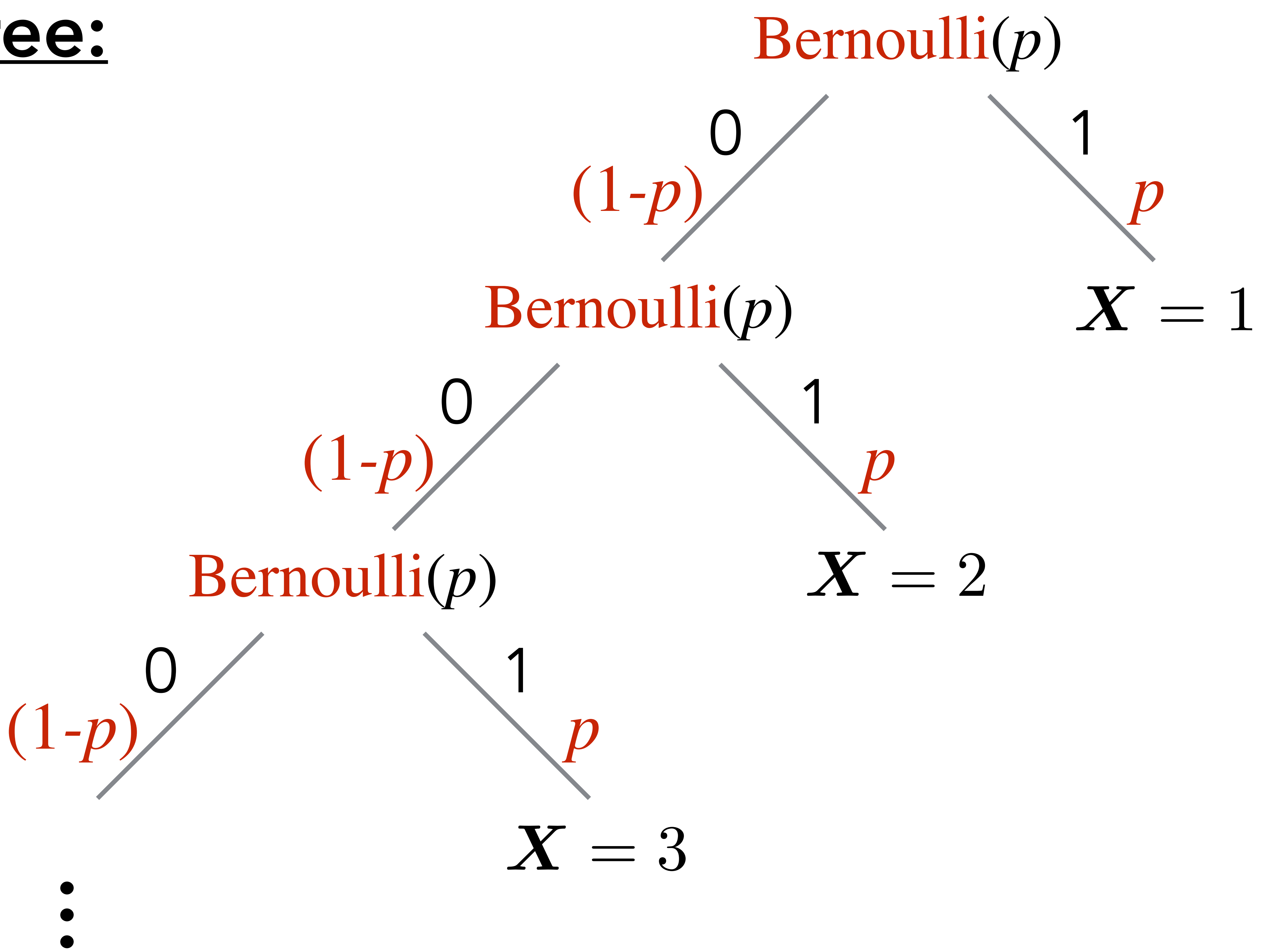
Tree:



Properties:

Geometric Random Variable

Tree:

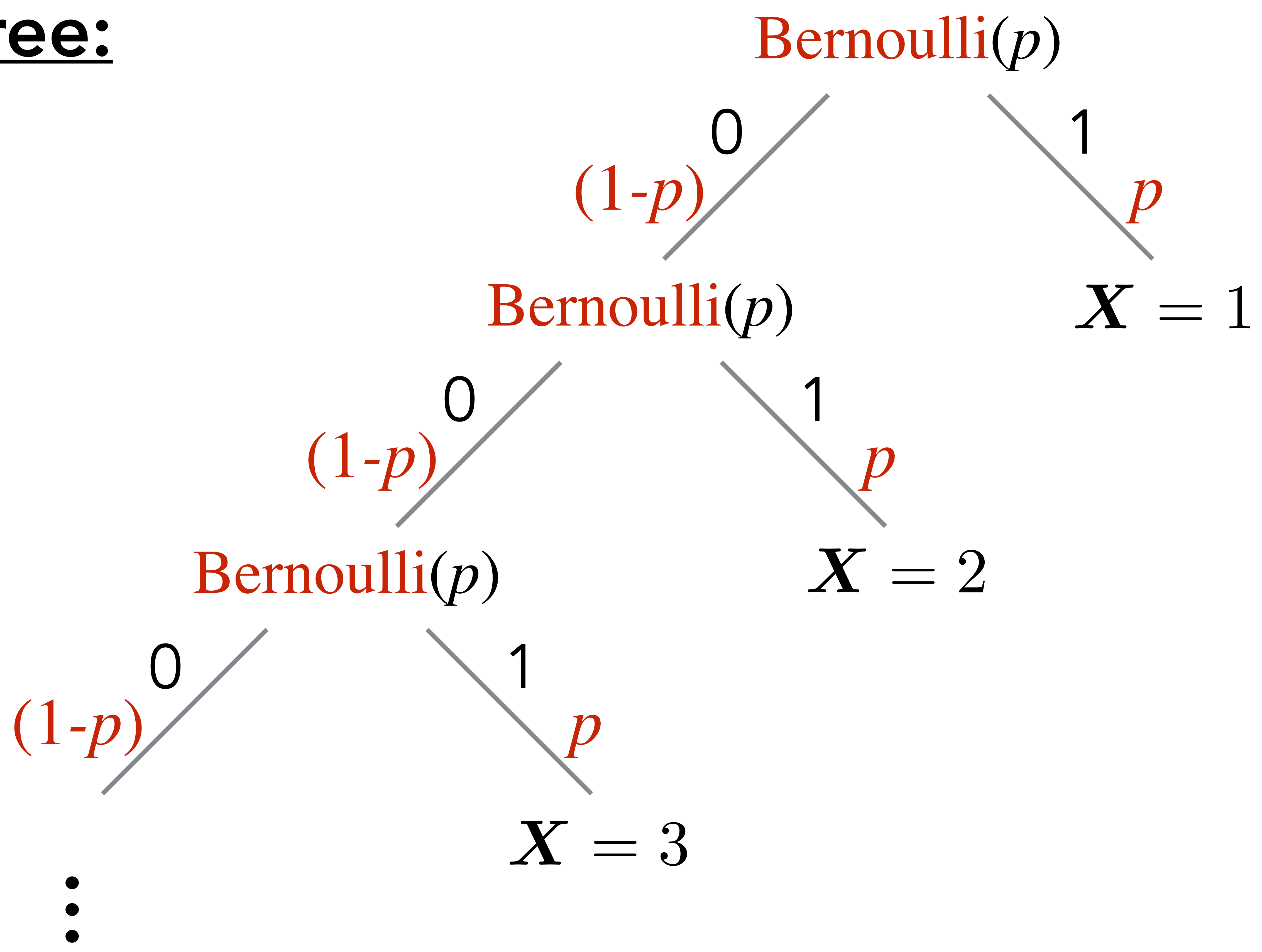


Properties:

$$\text{range}(\boldsymbol{X}) =$$

Geometric Random Variable

Tree:

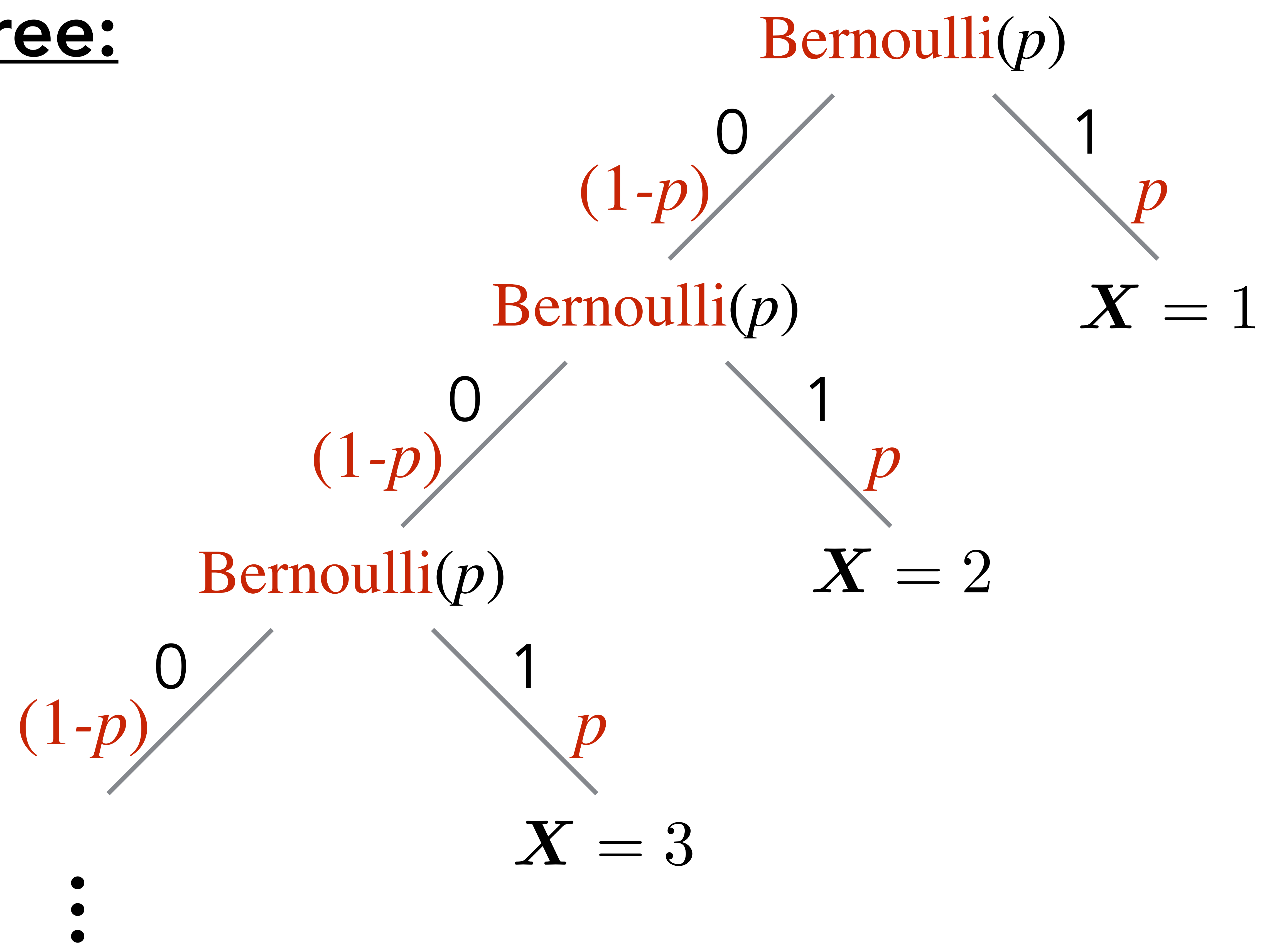


Properties:

$$\mathrm{range}(\boldsymbol{X}) = \{1, 2, 3, \ldots\}$$

Geometric Random Variable

Tree:



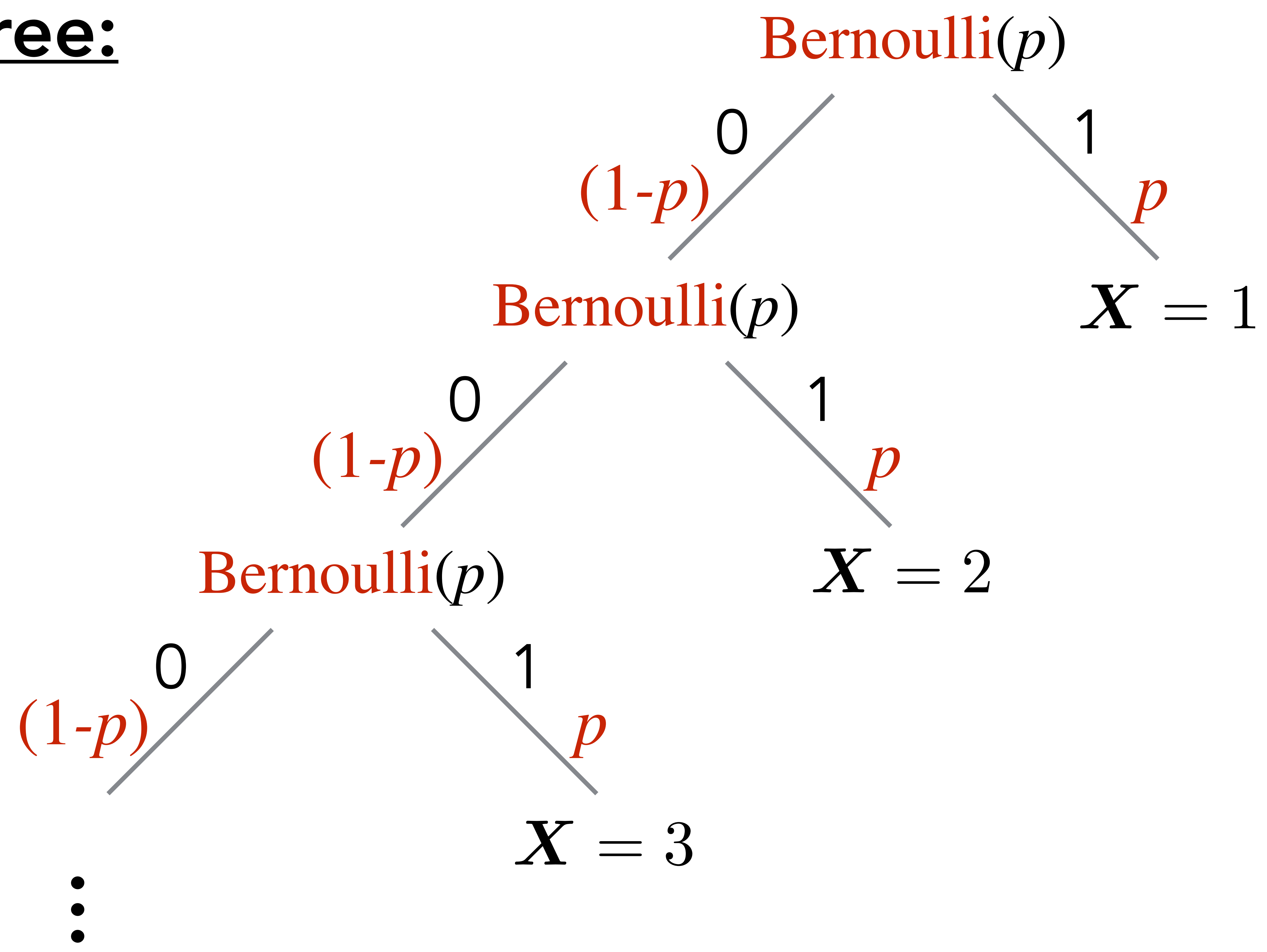
Properties:

$$\text{range}(\boldsymbol{X}) = \{1, 2, 3, \ldots\}$$

$$\Pr[\boldsymbol{X} = i] =$$

Geometric Random Variable

Tree:



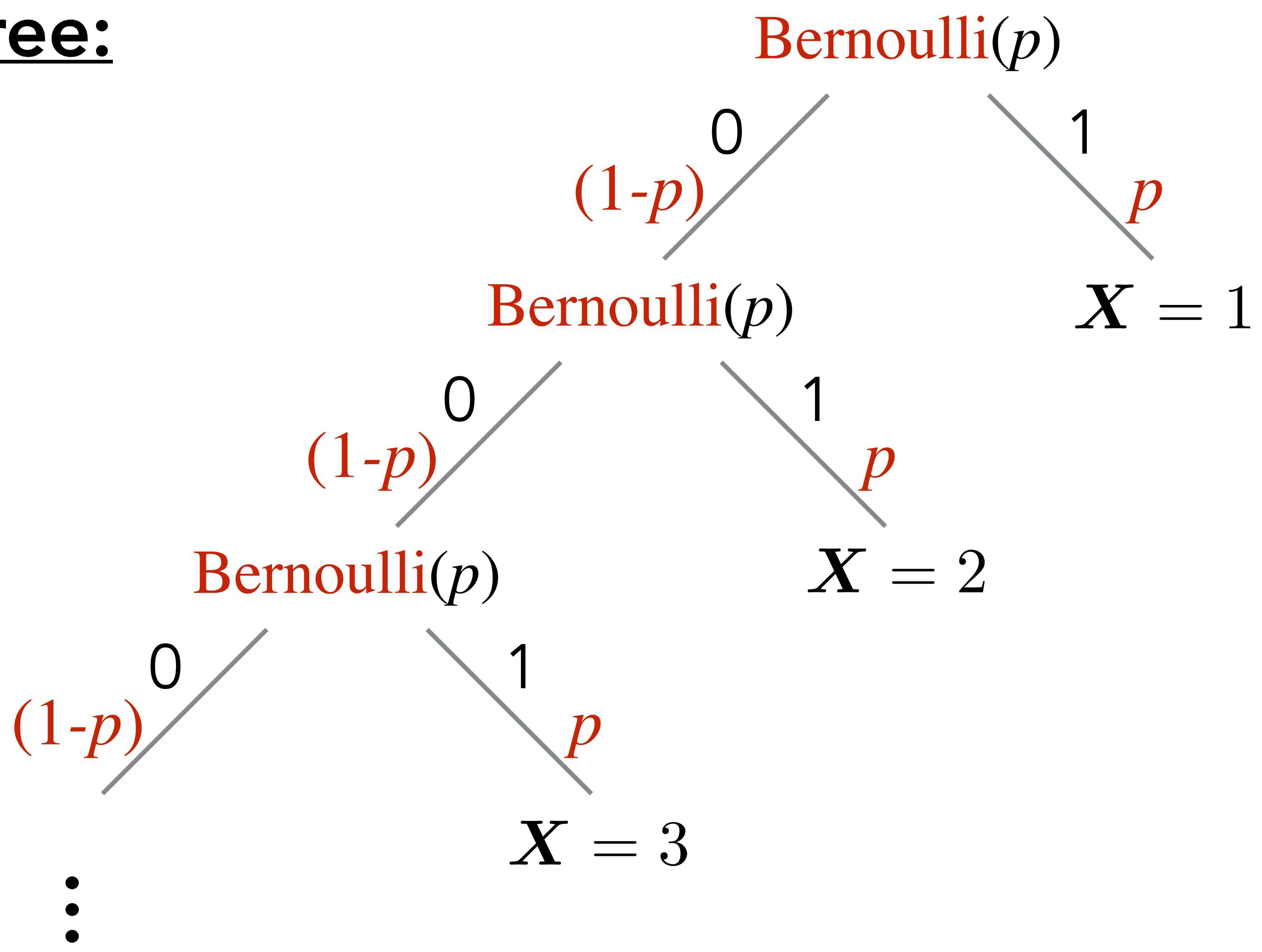
Properties:

$$\text{range}(\boldsymbol{X}) = \{1, 2, 3, \ldots\}$$

$$\Pr[\boldsymbol{X} = i] = (1-p)^{i-1}p$$

Geometric Random Variable

Tree:



Properties:

$$\text{range}(\boldsymbol{X}) = \{1, 2, 3, \ldots\}$$

$$\Pr[\boldsymbol{X} = i] = (1-p)^{i-1}p$$

$$\mathbf{E}[\boldsymbol{X}] = 1/p$$

Next Time:

Introduction to Randomized Algorithms