Uncountable Sets

1 Comparison of Sets

Definition (Injection, surjection, and bijection)**.** Let X and Y be two (possibly infinite) sets.

- A function $f : X \to Y$ is called *injective* if for any $x, x' \in X$ such that $x \neq x'$, we have $f(x) \neq f(x')$. We write $X \hookrightarrow Y$ if there exists an injective function from X to Y.
- A function $f : X \to Y$ is called *surjective* if for all $y \in Y$, there exists an $x \in X$ such that $f(x) = y$. We write $X \twoheadrightarrow Y$ if there exists a surjective function from X to Y.
- A function $f : X \to Y$ is called *bijective* (or *one-to-one correspondence*) if it is both injective and surjective. We write $X \leftrightarrow Y$ if there exists a bijective function from X to Y.

Theorem (Relationships between different types of functions)**.** *Let X, Y and Z be three (possibly infinite) sets. Then,*

- *$X \hookrightarrow Y$ if and only if $Y \twoheadrightarrow X$;*
- *if $X \hookrightarrow Y$ and $Y \hookrightarrow Z$, then $X \hookrightarrow Z$;*
- *$X \leftrightarrow Y$ if and only if $X \hookrightarrow Y$ and $Y \hookrightarrow X$.*

Definition (Comparison of cardinality of sets)**.** Let X and Y be two sets.

- We write $|X| = |Y|$ if $X \leftrightarrow Y$.
- We write $|X| \leq |Y|$ (or $|Y| \geq |X|$) if $X \hookrightarrow Y$, or equivalently, if $Y \twoheadrightarrow X$.
- We write $|X| < |Y|$ (or $|Y| > |X|$) if it is not the case that $|X| \geq |Y|$.

Remark (Sanity checks for comparing cardinality of sets)**.** Theorem (Relationships between different types of functions) justifies the use of the notation $=$, $\leq$, $\geq$, $<$ and $>$. The properties we would expect to hold for this type of notation indeed do hold. For example,

- $|X| = |Y|$ if and only if $|X| \leq |Y|$ and $|Y| \leq |X|$,
- if $|X| \leq |Y| \leq |Z|$, then $|X| \leq |Z|$,
- if $|X| \leq |Y| < |Z|$, then $|X| < |Z|$, and so on.

Note (Set inclusion vs cardinality)**.** If $X \subseteq Y$, then $|X| \leq |Y|$ since the identity function that maps $x \in X$ to $x \in Y$ is an injection.

Proposition ($|\mathbb{S}| = |\mathbb{N}|$)**.** *Let $\mathbb{S} = \{0, 1, 4, 9, \ldots\}$ be the set of squares. Then $|\mathbb{S}| = |\mathbb{N}|$.*

Proof. The function $f : \mathbb{N} \to \mathbb{S}$ defined as $f(n) = n^2$ is a bijection. It is an injection since if $n^2 = m^2$, then $n = m$. And it is surjective since for every $s \in \mathbb{S}$, by the definition of $\mathbb{S}$, there exists an n such that $n^2 = s$. □

Proposition ($|\mathbb{Z}| = |\mathbb{N}|$)**.** *Let $\mathbb{Z}$ be the set of integers. Then $|\mathbb{Z}| = |\mathbb{N}|$.*

Proof. Since $\mathbb{N} \subseteq \mathbb{Z}$, we have $|\mathbb{N}| \leq |\mathbb{Z}|$. So we just need to argue $|\mathbb{Z}| \leq |\mathbb{N}|$, i.e. there is a surjective function f from $\mathbb{N}$ to $\mathbb{Z}$. For this, our strategy is to argue that we can list the elements of $\mathbb{Z}$ such that every element eventually appears in the list. If we can do this, then we have the surjection f that we want: we let $f(n)$ be the n'th element in the list. Creating such a listing of $\mathbb{Z}$ is relatively straightforward:

$$0,\ 1,\ -1,\ 2,\ -2,\ 3,\ -3,\ \ldots$$

With this listing, the mapping f is such that odd numbers are mapped to positive integers and even numbers are mapped to non-positive integers. The formula for f is

$$f(n) = \begin{cases} -n/2 & \text{if } n \text{ is even,} \\ (n+1)/2 & \text{if } n \text{ is odd.} \end{cases}$$

□

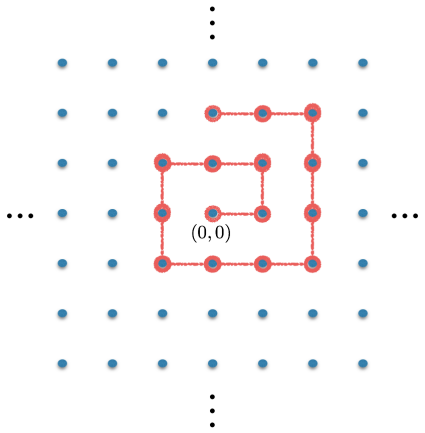
Proposition ($|\mathbb{Z} \times \mathbb{Z}| = |\mathbb{N}|$)**.** *Let $\mathbb{Z} \times \mathbb{Z}$ be the set of tuples with integer coordinates. Then $|\mathbb{Z} \times \mathbb{Z}| = |\mathbb{N}|$.*

Proof. We will show $|\mathbb{Z} \times \mathbb{Z}| = |\mathbb{N}|$ by arguing $|\mathbb{N}| \leq |\mathbb{Z} \times \mathbb{Z}|$ and $|\mathbb{Z} \times \mathbb{Z}| \leq |\mathbb{N}|$.

The former doesn't really require much of an argument. The map $n \mapsto (n, 0)$ is an injection and so $|\mathbb{N}| \leq |\mathbb{Z} \times \mathbb{Z}|$.

For the latter, we will show that there is a surjective function f from $\mathbb{N}$ to $\mathbb{Z} \times \mathbb{Z}$, and we will do so by arguing that we can list the elements of $\mathbb{Z} \times \mathbb{Z}$ such that every element eventually appears in the list. If we can do this, then we have the surjection f that we want: we let $f(n)$ be the n'th element in the list.

We now describe how to list the elements of $\mathbb{Z} \times \mathbb{Z}$. Consider the plot of $\mathbb{Z} \times \mathbb{Z}$ on a 2-dimensional grid. Starting at $(0, 0)$ we list the elements of $\mathbb{Z} \times \mathbb{Z}$ using a spiral shape, as shown below.



(The picture shows only a small part of the spiral.) Since we have a way to list all the elements such that every element eventually appears in the list, we are done.

(Side node: It is not a requirement that we give an explicit formula for $f(i)$. In fact, sometimes in such proofs, an explicit formula may not exist. This does not make the proof any less rigorous! The above proof is perfectly acceptable.) □

Proposition ($|\mathbb{Q}| = |\mathbb{N}|$)**.** *Let $\mathbb{Q}$ be the set of rational numbers. Then $|\mathbb{Q}| = |\mathbb{N}|$.*

Proof. We want to show $|\mathbb{Q}| = |\mathbb{N}|$, and it is clear that $|\mathbb{N}| \leq |\mathbb{Q}|$ (since $\mathbb{N} \subseteq \mathbb{Q}$), so we just need to show $|\mathbb{Q}| \leq |\mathbb{N}|$. We will make use of the previous proposition to establish this.

Note that every element of $\mathbb{Q}$ can be written as a fraction a/b where $a, b \in \mathbb{Z}$. In other words, there is a surjection from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Q}$ that maps (a, b) to a/b (if $b = 0$, map (a, b) to say 0). This shows that $|\mathbb{Q}| \leq |\mathbb{Z} \times \mathbb{Z}|$. From the previous proposition, $|\mathbb{Z} \times \mathbb{Z}| = |\mathbb{N}|$. Putting things together, $|\mathbb{Q}| \leq |\mathbb{N}|$. □

Proposition ($|\Sigma^*| = |\mathbb{N}|$)**.** *Let Σ be a finite non-empty set. Then $|\Sigma^*| = |\mathbb{N}|$.*

Proof. Recall that Σ^* denotes the set of all words/strings over the alphabet Σ with finitely many symbols. We want to show $|\mathbb{N}| \leq |\Sigma^*|$ and $|\Sigma^*| \leq |\mathbb{N}|$.

To show $|\mathbb{N}| \leq |\Sigma^*|$, note that the function that maps a string to its length, $s \mapsto |s|$, is a surjection from Σ^* to $\mathbb{N}$. (Remark: When $|\Sigma| = 1$, this function is a bijection.)

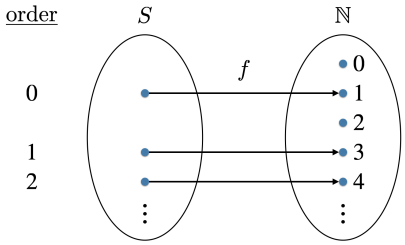
We will now show $|\Sigma^*| \leq |\mathbb{N}|$ by presenting a way to list all the elements of Σ^* such that eventually all the elements appear in the list. (As before, if we can present this listing, then we have a surjection $f : \mathbb{N} \to \Sigma^*$: we let $f(n)$ be the n'th element in the list.)

For each $k = 0, 1, 2, \ldots$, let Σ^k denote the set of words in Σ^* that have length exactly k. Note that Σ^k is a finite set for each k, and Σ^* is a union of these sets: $\Sigma^* = \Sigma^0 \cup \Sigma^1 \cup \Sigma^2 \cup \cdots$. This gives us a way to list the elements of Σ^* so that any element of Σ^* eventually appears in the list. First list the elements of Σ^0, then list the elements of Σ^1, then list the elements of Σ^2, and so on. □

Theorem (No cardinality strictly between finite and $|\mathbb{N}|$)**.** *If S is an infinite set and $|S| \leq |\mathbb{N}|$, then $|S| = |\mathbb{N}|$.*

Exercise (Proof of no cardinality strictly between finite and $|\mathbb{N}|$)**.** Prove the above theorem.

Solution. Assume S is such that $|S| \leq |\mathbb{N}|$ and S is infinite. Since $|S| \leq |\mathbb{N}|$, there is an injection $f : S \to \mathbb{N}$. This f allows us to define an ordering on S. For $s, t \in S$, write $s < t$ if $f(s) < f(t)$. Using this ordering, we can define the order of an element $s \in S$ as $\text{ord}(s) = |\{x \in S : f(x) < f(s)\}|$.



We now observe that the ord function that we have defined is a bijection from S to $\mathbb{N}$. To see that this is a bijection, first note that it is a surjective function because S is an infinite set. And it is an injective function because it is a total order. In particular, if $\text{ord}(s) = \text{ord}(s')$, then it must be $f(s) = f(s')$, and since f is injective, this implies $s = s'$. ■

Proposition ($|\mathbb{P}| = |\mathbb{N}|$)**.** *Let $\mathbb{P} = \{2, 3, 5, 7, \ldots\}$ be the set of prime numbers. Then $|\mathbb{P}| = |\mathbb{N}|$.*

Proof. Since $\mathbb{P} \subseteq \mathbb{N}$, we know $|\mathbb{P}| \leq |\mathbb{N}|$. We also know that there are infinitely many primes. Then by Theorem (No cardinality strictly between finite and $|\mathbb{N}|$), $|\mathbb{P}| = |\mathbb{N}|$.

(One can also prove this proposition without invoking the theorem. The function that maps n to the n'th prime number is a bijection.) □

2 Countable Sets

In the last section, we have shown $|\mathbb{N}| = |\mathbb{S}| = |\mathbb{Z}| = |\mathbb{P}| = |\mathbb{Z} \times \mathbb{Z}| = |\mathbb{Q}| = |\Sigma^*|$. In light of Theorem (No cardinality strictly between finite and $|\mathbb{N}|$), there are three categories of sets:

1. finite sets,
2. sets S such that $|S| = |X|$, where X is any of the sets listed above (like $\mathbb{N}$),
3. all other sets (i.e. sets S with $|S| > |X|$, where X is any of the sets listed above).

It makes sense to give a name for these different categories. In the literature, the first two categories combined is known as *countable sets*. And the third category is known as *uncountable* sets.

We can define a countable set as any set S with $|S| \leq |X|$, where $X = \mathbb{N}$. But the choice of $\mathbb{N}$ here, in a sense, is arbitrary. We can also choose, for instance, $X = \Sigma^*$, in which case we see countability is equivalent to encodability. And arguably, encodability captures the essence of this class of sets better. That being said, in the literature, $X = \mathbb{N}$ is the standard choice, leading to the definition below.

Definition (Countable and uncountable sets)**.**

- A set S is called *countable* if $|S| \leq |\mathbb{N}|$.
- A set S is called *countably infinite* if it is countable and infinite.
- A set S is called *uncountable* if it is not countable, i.e. $|S| > |\mathbb{N}|$.

Note (Only two options for countable sets)**.** Theorem (No cardinality strictly between finite and $|\mathbb{N}|$) implies that if S is countable, there are two options: either S is finite, or $|S| = |\mathbb{N}|$.

Important (Countability is equivalent to encodability)**.** Recall that a set S is encodable if for some alphabet Σ, there is an injection from S to Σ^*, or equivalently, $|S| \leq |\Sigma^*|$ (Definition (**??**)). Since $|\mathbb{N}| = |\Sigma^*|$, we see that a set is countable if and only if it is encodable. Therefore, one can show that a set is countable by showing that it is encodable. We will call this the "CS method" for showing countability.

The standard definition of countability ($|S| \leq |\mathbb{N}|$) highlights the following heuristic.

- If you can list the elements of S in a way that every element appears in the list eventually, then S is countable.

The encodability definition highlights another heuristic that is far more relevant in computer science.

- If you can "write down" each element of S using a finite number of symbols, then S is countable.

Proposition (The set of polynomials with rational coefficients is countable)**.** *The set of all polynomials in one variable with rational coefficients is countable.*

Proof. Let $\mathbb{Q}[x]$ denote the set of all polynomials in one variable with rational coefficients. We want to show that $\mathbb{Q}[x]$ is countable. We will do so using the CS method. Let

$$\Sigma = \{\texttt{0}, \texttt{1}, \texttt{2}, \texttt{3}, \texttt{4}, \texttt{5}, \texttt{6}, \texttt{7}, \texttt{8}, \texttt{9}, \texttt{+}, \texttt{-}, \texttt{/}, \texttt{x}\}.$$

Then observe that every element of $\mathbb{Q}[x]$ can be written as a finite string over this alphabet. For example,

`2x3-1/34x2+99/100x1+22/7`

represents the polynomial

$$2x^3 - 1/34x^2 + 99/100x + 22/7.$$

This implies that there is a surjective map from Σ^* to $\mathbb{Q}[x]$. And therefore $|\mathbb{Q}[x]| \leq |\Sigma^*|$. Since Σ^* is countable, i.e. $|\Sigma^*| \leq |\mathbb{N}|$, $\mathbb{Q}[x]$ is also countable. □

Exercise (Practice with countability proofs)**.** Show that the following sets are countable.

(a) $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$.

(b) The set of all functions $f : S \to \mathbb{N}$, where S is some fixed finite set.

Solution. Part 1: We want to show that $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ is countable. We use the CS method with $\Sigma = \{\texttt{0}, \texttt{1}, \texttt{2}, \texttt{3}, \texttt{4}, \texttt{5}, \texttt{6}, \texttt{7}, \texttt{8}, \texttt{9}, \texttt{-}, \texttt{\$}\}$. Note that any element of $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ can be written uniquely as a finite word over Σ (we use the dollar sign as a separator between the integers). As an illustration, $(9234851, \text{-}1234, 0) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ can be encoded as the string `9234851$-1234$0`. Every integer has finite length, so the string encoding is always of finite length.
Part 2: Let U be the set of all functions $f : S \to \mathbb{N}$, where S is a finite set. We want to show that U is countable.

We first make an observation about the elements of U. Take a function $f : S \to \mathbb{N}$, where S is a finite set. Let k be the size of S and let $s_1, s_2, \ldots, s_k$ be its elements. Then f can be uniquely represented by the tuple

$$(f(s_1), f(s_2), \ldots, f(s_k)),$$

where each element of the tuple is an element from $\mathbb{N}$.

We now show that U is countable using the CS method with the alphabet $\Sigma = \{\texttt{0}, \texttt{1}, \texttt{2}, \texttt{3}, \texttt{4}, \texttt{5}, \texttt{6}, \texttt{7}, \texttt{8}, \texttt{9}, \texttt{\$}\}$. The observation above shows that any element of U can be uniquely represented with a finite length string where commas are replaced with `$`. (Note that there is no need to put the opening and closing parentheses.) This suffices to conclude that U is countable. ■

3 Diagonalization and Uncountable Sets

In this section we explain the diagonalization proof technique, which is one of the most powerful techniques in mathematics.

The basic general question that diagonalization is trying to answer is the following.

> Given a set of objects $\mathcal{F}$ part of a larger universe $\mathcal{U}$ (so $\mathcal{F} \subseteq \mathcal{U}$), is $\mathcal{F} = \mathcal{U}$? And if $\mathcal{F} \neq \mathcal{U}$, can we construct an element $D \in \mathcal{U} \setminus \mathcal{F}$?

Below are some examples of fundamental questions of this flavor. Diagonalization can be used to answer all of them.

- Let $\mathcal{F}$ be the set of rational numbers and $\mathcal{U}$ be the set of reals. Is there an irrational number, and if there is, can we construct one?
- It is not hard to give a direct proof that $\sqrt{2}$ is irrational, so the above question may not be terribly interesting. Here is a much more non-trivial question. Let $\mathcal{F}$ be the set of real algebraic numbers, and $\mathcal{U}$ be the set of reals. Is there a real non-algebraic number (i.e. is there a real transcendental number)? And if there is, can we construct one?
- Let $\mathcal{F}$ be the set of all decidable languages, and let $\mathcal{U}$ be the set of all languages. Is there an undecidable language, and if there is, can we construct one?
- Let $\mathcal{F}$ be the set of all languages that can be solved in at most n^k time, and let $\mathcal{U}$ be the set of all languages that can be solved in at most n^{k+1} time (we will cover time complexity in a future chapter). Is there a language in $\mathcal{U}$ that is not in $\mathcal{F}$, and if there is, can we construct one?
- Let $\mathcal{F}$ be the set of all provable statements and let $\mathcal{U}$ be the set of all true statements. Is there a true statement that is not provable?

Even though all these questions have the same basic structure, they are quite different since they involve different types of objects. In order to identify a general technique that can be applied in these kinds of different settings, we'll express diagonalization as a statement involving *functions*, because many mathematical objects can be conveniently viewed as functions.

Note (Functions in disguise)**.** The following are some examples of how various mathematical objects have a natural representation as a function.

- **Sets.** A set $S \subseteq X$ can be viewed as a function $f_S : X \to \{0, 1\}$, where $f_S(x) = 1$ if and only if $x \in S$. This is called the *characteristic function* of the set. Recall that we made this observation in Important Note (**??**).
- **Finite sequences.** A sequence s of length k with elements from a set Y can be viewed as a function $f_s : \{0, 1, 2, \ldots, k-1\} \to Y$, where $f_s(i)$ is the i'th element of the sequence (starting counting from 0). In other words, given f_s, the corresponding sequence is $(f_s(0), f_s(1), \ldots, f_s(k-1))$.
- **Infinite sequences.** Similarly, an infinite-length sequence s with elements from Y can be viewed as a function $f_s : \mathbb{N} \to Y$.
- **Numbers.** Numbers can be viewed as functions since the binary representation of a number is a sequence of bits (possibly infinite-length).

  For example, all real numbers between 0 and 1 (inclusive) is denoted by $[0, 1]$. Every function $f : \mathbb{N} \to \{0, 1\}$ represents a real number in $[0, 1]$ in binary, namely $0.f(0)f(1)f(2)\ldots$.

Lemma (Diagonalization)**.** *Let $\mathcal{F}$ be a set of functions $f : X \to Y$ where $|Y| \geq 2$. If $|X| \geq |\mathcal{F}|$, we can construct a function $f_D : X \to Y$ such that $f_D \notin \mathcal{F}$.*

Proof. Given a set $\mathcal{F}$ of functions $f : X \to Y$, we want to construct a function $f_D : X \to Y$ that is different from every $f \in \mathcal{F}$. The main idea is the following. For each $f \in \mathcal{F}$, pick a unique input $x \in X$ and define $f_D(x)$ in a way such that it is different from $f(x)$. Since by construction f_D and f disagree on input x, f_D is different from f. And since we do this for every $f \in \mathcal{F}$, f_D is different from all $f \in \mathcal{F}$.

Above, it is important that we pick a unique x for each $f \in \mathcal{F}$ so that f_D can be defined in a consistent way. The ability to pick a unique x for each $f \in \mathcal{F}$ is equivalent to $|X| \geq |\mathcal{F}|$.

A bit more formally, since $|X| \geq |\mathcal{F}|$, there is an injection $\phi : \mathcal{F} \to X$. Let $x_f = \phi(f)$. So $f \neq f'$ implies $x_f \neq x_{f'}$. Define f_D such that for all $f \in \mathcal{F}$, $f_D(x_f) \neq f(x_f)$ (this is where the assumption $|Y| \geq 2$ is used). This ensures that f_D is different from every $f \in \mathcal{F}$, and therefore $f_D \notin \mathcal{F}$. (Note that our description of f_D leaves it underspecified, but see the remark below.) □

Remark (There are many choices for the diagonal element)**.** When we apply the above lemma to construct an explicit $f_D \notin \mathcal{F}$, we call that *diagonalizing against the set $\mathcal{F}$*. And we call f_D a *diagonal element*. Typically there are several choices for the definition of f_D:

- Different injections $\phi : \mathcal{F} \to X$ can lead to different diagonal elements.
- If $|Y| > 2$, we have more than one choice on what value we assign to $f_D(x_f)$ that makes $f_D(x_f) \neq f(x_f)$ (here x_f denotes $\phi(f)$).
- If there are elements $x \in X$ not in the range of ϕ, then we can define $f_D(x)$ any way we like.

Important (Diagonalization gives an explicit construction)**.** Note that diagonalizing against a set $\mathcal{F}$ produces an explicit function f_D not in $\mathcal{F}$. Therefore, in situations where you wish to find an explicit object that is not in a given set, you should consider if diagonalization could be applied.

Corollary (Direct corollary of diagonalization)**.** *If $\mathcal{F}$ is the set of **all** functions $f : X \to Y$ (and $|Y| \geq 2$), then $|X| < |\mathcal{F}|$.*

Proof. Diagonalization tells us that whenever $|X| \geq |\mathcal{F}|$, we can construct a function f_D not in $\mathcal{F}$. But if $\mathcal{F}$ denotes the set of all functions $f : X \to Y$, the construction of f_D is not possible. Therefore, it must be the case that $|X| < |\mathcal{F}|$. □

Exercise (Diagonalization and uncountability)**.** Using the corollary above, give an example of an uncountable set.

Solution. Let $\mathcal{F}$ be the set of all functions $f : \mathbb{N} \to \{0, 1\}$. Then by the above corollary, $|\mathbb{N}| < |\mathcal{F}|$, and therefore $\mathcal{F}$ is uncountable. ■

Theorem (Cantor's Theorem)**.** *For any set X, $|X| < |\wp(X)|$.*

Proof. Let $\mathcal{F}$ be the set of all functions $f : X \to \{0, 1\}$, which are the characteristic functions of the subsets of X (so $f(x) = 1$ if and only if $x \in X$). Then by Note (Functions in disguise), $|\mathcal{F}| = |\wp(X)|$ and the result follows from the above Corollary. □

Remark (Russell's Paradox as diagonalization)**.** In the famous Russell's Paradox, we consider the set of all sets that do not contain themselves. That is, we consider

$$D = \{\text{set } S : S \notin S\}.$$

Then we ask whether D is in D or not. If $D \in D$, then by the definition of D, $D \notin D$. And if $D \notin D$, again by the definition of D, $D \in D$. Either way we get a contradiction. Therefore, the conclusion is that such a set D should not be mathematically definable.

This paradox is also an application of diagonalization. For any set X, we cannot diagonalize against the set of *all* functions $f : X \to \{0, 1\}$. If we imagine diagonalizing against this set, where X is the set of all sets, the natural diagonal element f_D that comes out of diagonalization is precisely the characteristic function of the set D defined above.

Corollary (The power set of an infinite set is uncountable)**.** *For any infinite set S, $\wp(S)$ is uncountable. In particular, $\wp(\mathbb{N})$ is uncountable.*

Proof. Let's break this up into two cases: (i) S is countably infinite and (ii) S is uncountable.

If S is countably infinite, $|S| = |\mathbb{N}|$. And by Cantor's Theorem, $|\mathbb{N}| = |S| < |\wp(S)|$, which by definition means $\wp(S)$ is uncountable.

If S is uncountable, then $|\mathbb{N}| < |S| < |\wp(S)|$, and so once again, $\wp(S)$ is uncountable. □

Note (Diagonalization with countable sets)**.** In a common scenario where diagonalization is applied, both $\mathcal{F}$ and X are countably infinite sets. So we can list the elements of $\mathcal{F}$ as

$$f_1,\ f_2,\ f_3,\ \ldots$$

as well as the elements of X as

$$x_1,\ x_2,\ x_3,\ \ldots$$

Then for all i, define $f_D(x_i)$ such that $f_D(x_i) \neq f_i(x_i)$. If $Y = \{0, 1\}$, for example, $f_D(x_i) = \textbf{not } f_i(x_i)$. The construction of f_D can be nicely visualized with a table, as shown below. Here, an entry corresponding to row f_i and column x_j contains the value $f_i(x_j)$.

	x_1	x_2	x_3	$\cdots$
f_1	$\boxed{0}$	1	0	
f_2	1	$\boxed{1}$	1	$\cdots$
f_3	1	1	$\boxed{0}$	
$\vdots$		$\vdots$		
f_D	1	0	1	$\cdots$

By construction, the diagonal element f_D differs from every f_i, $i \in \{1, 2, 3, \ldots\}$. In particular, it differs from f_i with respect to the input x_i.

Below, we will apply Lemma (Diagonalization) to construct an irrational number (i.e. a number in $\mathbb{R} \setminus \mathbb{Q}$). Even though the result may not be interesting (since there is a relatively simple proof that $\sqrt{2}$ is irrational), it does illustrate the use of diagonalization nicely. Cantor used the same technique to construct an explicit transcendental number, which was a quite non-trivial and important result at the time. Instead of proving that result, we put it in the practice set for you.

Proposition (Irrational numbers exist)**.** *There exists an irrational real number, that is, there exists a number in* $\mathbb{R} \setminus \mathbb{Q}$.

Proof. We will prove the existence of a number in $\mathbb{R} \setminus \mathbb{Q}$ (i.e. the existence of an irrational number) by constructing such a number using diagonalization. In order to simplify the proof, we will restrict our attention to numbers in the interval $[0, 1]$, so we will construct an irrational number in $[0, 1]$. The restriction to $[0, 1]$ allows us to represent a number just by considering the fractional part.

As mentioned in Note (Functions in disguise), numbers can be viewed as functions: Every function $f : \mathbb{N}^+ \to \{0, 1\}$ represents a real number in $[0, 1]$ in binary, namely $0.f(1)f(2)f(3)\ldots$. Two different functions f and f' may represent the same real number, e.g. $0.10000\ldots$ and $0.01111\ldots$ represent the same real number. But we don't have more than two different functions representing the same number.

Now consider the set $\mathcal{F}$ of all functions $f : \mathbb{N}^+ \to \{0, 1\}$ representing a rational number. We will diagonalize against $\mathcal{F}$ to construct f_D not in $\mathcal{F}$. This f_D then represents a real number that is not rational, and we are done.

In order to diagonalize against $\mathcal{F}$, all we need is that $|\mathbb{N}^+| \geq |\mathcal{F}|$ holds. And this is indeed the case. $\mathcal{F}$ is countably infinite because $\mathbb{Q}$ is countably infinite. And obviously $\mathbb{N}^+$ is countably infinite. This means we are in the situation outlined in Note (Diagonalization with countable sets), and $|\mathbb{N}^+| = |\mathcal{F}|$.

	1	2	3	$\cdots$
f_1	0	1	0	
f_2	1	1	1	$\cdots$
f_3	1	1	0	
$\vdots$		$\vdots$		
f_D	1	0	1	$\cdots$

□

Exercise ($\mathbb{R}$ is uncountable)**.** Prove that $\mathbb{R}$ is uncountable.

Solution. The proof is basically the same as the proof of Proposition (Irrational numbers exist). There, we have implicitly proved that the interval $[0, 1]$ cannot be countable. To see this, note that if $[0, 1]$ is countable, then the set $\mathcal{F}$ of **all** functions $f : \mathbb{N}^+ \to \{0, 1\}$ is countable. So $|\mathcal{F}| = |\mathbb{N}^+|$, which allows us to diagonalize against $\mathcal{F}$, which is not possible. ■

Definition (Σ^∞)**.** Let Σ be some finite alphabet. We denote by Σ^∞ the set of all *infinite* length words over the alphabet Σ.

Remark. Observe that $\Sigma^* \cap \Sigma^\infty = \varnothing$.

Theorem ($\{0,1\}^\infty$ is uncountable)**.** *The set* $\{0, 1\}^\infty$ *is uncountable.*

Proof. Using the observation in Note (Functions in disguise), an infinite-length string $s \in \{0,1\}^\infty$ corresponds to a function $f_s : \mathbb{N}^+ \to \{0,1\}$, which is the characteristic function of a set $S \subseteq \mathbb{N}^+$ (where $n \in S$ if and only if $f_s(n) = 1$). Therefore the set of all infinite-length strings, $\{0,1\}^\infty$, corresponds to the set of all subsets of $\mathbb{N}^+$, $\wp(\mathbb{N}^+)$. That is, $\{0,1\}^\infty \leftrightarrow \wp(\mathbb{N}^+)$. Using Corollary (The power set of an infinite set is uncountable) we can conclude $\{0,1\}^\infty$ is uncountable. □

Exercise (Uncountable sets are closed under supersets)**.** Prove that if X is uncountable and $X \subseteq Y$, then Y is also uncountable.

Solution. We want to show that if Y is a superset of an uncountable set X, then Y must be uncountable.

If X is uncountable, by definition, $|X| > |\mathbb{N}|$. If $X \subseteq Y$, then there is a clear injection from X to Y (map $x \in X$ to $x \in Y$), so $|X| \leq |Y|$. Combining this with $|X| > |\mathbb{N}|$, we have $|Y| \geq |X| > |\mathbb{N}|$, and therefore Y is uncountable. ■

Important (Uncountability via $\{0,1\}^\infty$)**.** If we want to show that a set X is uncountable, it suffices to show that $|X| \geq |Y|$ for some uncountable set Y. Typically, a good choice for such a Y is $\{0,1\}^\infty$. And one strategy for establishing $|X| \geq |\{0,1\}^\infty|$ is to identify a subset S of X such that $S \leftrightarrow \{0,1\}^\infty$.

Exercise (Practice with uncountability proofs)**.** Show that the following sets are uncountable.

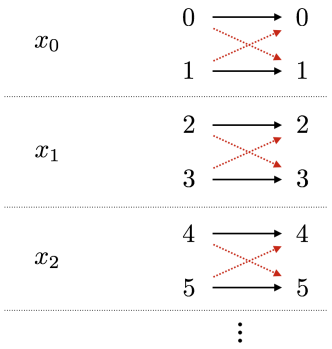
1. The set of all bijective functions from $\mathbb{N}$ to $\mathbb{N}$.
2. $\{x_1x_2x_3\ldots \in \{1,2\}^\infty : \text{ for all } n \geq 1,\ \sum_{i=1}^{n} x_i \not\equiv 0 \mod 4\}$

Solution. Part 1: Let S be the set of all bijective functions from $\mathbb{N}$ to $\mathbb{N}$. We want to show that S is uncountable, and we will do so by showing that $\{0,1\}^\infty \hookrightarrow S$, establishing $|\{0,1\}^\infty| \leq |S|$.

We now describe this injective mapping. Given $x \in \{0,1\}^\infty$, we map it to a bijection $f_x : \mathbb{N} \to \mathbb{N}$ as follows. Let x_n be the n'th bit of x, and assume the indexing starts from 0. Then for all $n \in \mathbb{N}$,

$$\begin{aligned} &\text{if } x_n = 0, \quad f_x \text{ maps } 2n \text{ to } 2n \text{ and } 2n+1 \text{ to } 2n+1; \\ &\text{if } x_n = 1, \quad f_x \text{ maps } 2n \text{ to } 2n+1 \text{ and } 2n+1 \text{ to } 2n. \end{aligned}$$

The below picture illustrates the construction of f_x. If $x_n = 0$, we pick the black arrows to map $2n$ and $2n+1$, and if $x_n = 1$ we pick the red/dashed arrows to map $2n$ and $2n+1$.



Observe that for any $x \in \{0,1\}^\infty$, the corresponding function f_x is indeed a bijection. It is also clear that if $x \neq x'$, then $f_x \neq f_{x'}$. So this mapping from $\{0,1\}^\infty$ to S is indeed an injection. This completes the proof.

Part 2: Let $S = \{x_1x_2x_3\ldots \in \{1,2\}^\infty : \text{ for all } n \geq 1,\ \sum_{i=1}^{n} x_i \not\equiv 0 \mod 4\}$. We want to show that S is uncountable, and we will do so by identifying a subset of S that is in one-to-one correspondence with $\{0,1\}^\infty$.

Let $a = 22$ and $b = 112$. Define the set $S' = \{1w : w \in \{a,b\}^\infty\}$. There are a couple of important observations (whose proofs are omitted):

- $S' \subseteq S$.

- The mapping $f : \{0,1\}^\infty \to \{a,b\}^\infty$ such that $y_1 y_2 \ldots \in \{0,1\}^\infty$ is mapped to $w_1 w_2 \ldots \in \{a,b\}^\infty$, where $w_i = a$ if $y_i = 0$ and $w_i = b$ if $y_i = 1$, is a bijection.

These two observations imply that we have identified a subset of S (namely S') that is in one-to-one correspondence with $\{0,1\}^\infty$, which allows us to conclude that S is uncountable. ■

4 Check Your Understanding

Problem. 1. For sets X, Y, what are the definitions of $|X| \leq |Y|$, $|X| \geq |Y|$, $|X| = |Y|$, and $|X| < |Y|$?

2. What is the definition of a countable set?
3. What is the CS method for showing that a set is countable?
4. True or false: There exists an infinite set S such that $|S| < |\mathbb{N}|$.
5. True or false: $\{0,1\}^* \cap \{0,1\}^\infty = \varnothing$.
6. True or false: $|\{0,1,2\}^*| = |\mathbb{Q} \times \mathbb{Q}|$.
7. State the Diagonalization Lemma and briefly explain how it is proved.
8. What is the connection between Diagonalization Lemma and uncountability?
9. What is Cantor's Theorem?
10. True or false: $|\wp(\{0,1\}^\infty)| = |\wp(\wp(\{0,1\}^\infty))|$.
11. True or false: There is a surjection from $\{0,1\}^\infty$ to $\{0,1,2,3\}^\infty$.
12. True or false: Let Σ be an alphabet. The set of encodings mapping $\mathbb{N}$ to Σ^* is countable.
13. True or false: Let $\Sigma = \{1\}$ be a unary alphabet. The set of all languages over Σ is countable.
14. State a technique for proving that a given set is uncountable.

5 High-Order Bits

Important. Here are the important things to keep in mind from this chapter.

1. The definitions of injective, surjective and bijective functions are fundamental.
2. The concepts of countable and uncountable sets, and their precise mathematical definitions.
3. When it comes to showing that a set is countable, the CS method is the best choice, almost always.
4. The Diagonalization Lemma is one of the most important and powerful techniques in all mathematics.
5. When showing that a set is uncountable, establishing an injection from $\{0,1\}^\infty$ (or a surjection to $\{0,1\}^\infty$) is one of the best techniques you can use.